量 子 场 论

第 9 章　分立对称性和 Majorana 旋量场

9.6 节和 9.7 节

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9.6 节　Weyl、Dirac 和 Majorana 旋量

9.6.1 小节　左手和右手 Weyl 旋量

- **Dirac 旋量场**和 **Majorana 旋量场**都可以**分解**为**左手**和**右手**的 Weyl 旋量场
- 为了更深刻地认识旋量场，本节进一步研究 **Weyl 旋量**
- 用 $\sigma^\mu = (\mathbf{1}, \boldsymbol{\sigma})$ 和 $\bar{\sigma}^\mu = (\mathbf{1}, -\boldsymbol{\sigma})$ 定义 **2 × 2 矩阵** $s^{\mu\nu} \equiv \frac{\mathrm{i}}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$
- 由 $(\sigma^\mu)^\dagger = \sigma^\mu$ 和 $(\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu$ 推出

$$(s^{\mu\nu})^\dagger = -\frac{\mathrm{i}}{4}[(\bar{\sigma}^\nu)^\dagger(\sigma^\mu)^\dagger - (\bar{\sigma}^\mu)^\dagger(\sigma^\nu)^\dagger] = -\frac{\mathrm{i}}{4}(\bar{\sigma}^\nu\sigma^\mu - \bar{\sigma}^\mu\sigma^\nu) = \frac{\mathrm{i}}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$$

- 从而将 **Weyl 表象**中的**旋量表示生成元**化为

$$\mathcal{S}^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu] = \frac{\mathrm{i}}{4}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} = \begin{pmatrix} s^{\mu\nu} & \\ & (s^{\mu\nu})^\dagger \end{pmatrix}$$

- 也就是说，**4 × 4 矩阵** $\mathcal{S}^{\mu\nu}$ 是 2×2 矩阵 $s^{\mu\nu}$ 和 $(s^{\mu\nu})^\dagger$ 的**直和**
- 因而 $s^{\mu\nu}$ 和 $(s^{\mu\nu})^\dagger$ 是两个 Lorentz 群 **2 维表示**的**生成元**

左手和右手 Weyl 旋量所处 2 维表示

- 对于 Lorentz 变换 Λ 的一组**变换参数** $\omega_{\mu\nu}$，用 $s^{\mu\nu}$ 生成固有保时向有限变换

$$d(\Lambda) \equiv \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}s^{\mu\nu}\right)$$

- 它属于**左手 Weyl 旋量**所处的 **2 维表示**

- 相应的逆变换矩阵为 $d^{-1}(\Lambda) = \exp\left(\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}s^{\mu\nu}\right)$，取厄米共轭，得

$$d^{-1\dagger}(\Lambda) = \exp\left[-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}(s^{\mu\nu})^\dagger\right]$$

- 这是用 $(s^{\mu\nu})^\dagger$ 生成的固有保时向有限变换，属于**右手 Weyl 旋量**所处的 **2 维表示**

左手和右手 Weyl 旋量所处 2 维表示

对于 **Lorentz 变换** Λ 的一组**变换参数** $\omega_{\mu\nu}$，用 $s^{\mu\nu}$ 生成固有保时向有限变换

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这是用 $(s^{\mu\nu})^\dagger$ 生成的固有保时向有限变换，属于**右手 Weyl 旋量**所处的 **2 维表示**

于是，**旋量表示的 4×4 Lorentz 变换矩阵**分解为

$$D(\Lambda) = \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}S^{\mu\nu}\right) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\omega_{\mu\nu}s^{\mu\nu}/2} & \\ & \mathrm{e}^{-\mathrm{i}\omega_{\mu\nu}(s^{\mu\nu})^\dagger/2} \end{pmatrix} = \begin{pmatrix} d(\Lambda) & \\ & d^{-1\dagger}(\Lambda) \end{pmatrix}$$

因此，4 维旋量表示 $\{D(\Lambda)\}$ 是 2 维表示 $\{d(\Lambda)\}$ 和 $\{d^{-1\dagger}(\Lambda)\}$ 的**直和**

等价表示

利用 $\sigma^2\sigma^\mu\sigma^2=(\bar\sigma^\mu)^{\mathrm{T}}$ 和 $\sigma^2\bar\sigma^\mu\sigma^2=(\sigma^\mu)^{\mathrm{T}}$ 推出

$$\begin{aligned}
\sigma^2 s^{\mu\nu}\sigma^2 &= \frac{\mathrm{i}}{4}(\sigma^2\sigma^\mu\sigma^2\sigma^2\bar\sigma^\nu\sigma^2-\sigma^2\sigma^\nu\sigma^2\sigma^2\bar\sigma^\mu\sigma^2)\\
&= \frac{\mathrm{i}}{4}[(\bar\sigma^\mu)^{\mathrm{T}}(\sigma^\nu)^{\mathrm{T}}-(\bar\sigma^\nu)^{\mathrm{T}}(\sigma^\mu)^{\mathrm{T}}]=-(s^{\mu\nu})^{\mathrm{T}}\\
\sigma^2 d(\Lambda)\sigma^2 &= \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\sigma^2 s^{\mu\nu}\sigma^2\right)=\exp\left[\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}(s^{\mu\nu})^{\mathrm{T}}\right]\\
&= \left[\exp\left(\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}s^{\mu\nu}\right)\right]^{\mathrm{T}}=d^{-1\mathrm{T}}(\Lambda)
\end{aligned}$$

这里 $d^{-1\mathrm{T}}(\Lambda)=[d^{-1\dagger}(\Lambda)]^*$，线性表示 $\{d^{-1\mathrm{T}}(\Lambda)\}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ 的**复共轭表示**

等价表示

- 利用 $\sigma^2\sigma^\mu\sigma^2 = (\bar{\sigma}^\mu)^{\mathrm{T}}$ 和 $\sigma^2\bar{\sigma}^\mu\sigma^2 = (\sigma^\mu)^{\mathrm{T}}$ 推出

$$\begin{aligned}
\sigma^2 s^{\mu\nu}\sigma^2 &= \frac{\mathrm{i}}{4}(\sigma^2\sigma^\mu\sigma^2\sigma^2\bar{\sigma}^\nu\sigma^2 - \sigma^2\sigma^\nu\sigma^2\sigma^2\bar{\sigma}^\mu\sigma^2)\\
&= \frac{\mathrm{i}}{4}[(\bar{\sigma}^\mu)^{\mathrm{T}}(\sigma^\nu)^{\mathrm{T}} - (\bar{\sigma}^\nu)^{\mathrm{T}}(\sigma^\mu)^{\mathrm{T}}] = -(s^{\mu\nu})^{\mathrm{T}}\\
\sigma^2 d(\Lambda)\sigma^2 &= \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\sigma^2 s^{\mu\nu}\sigma^2\right) = \exp\left[\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}(s^{\mu\nu})^{\mathrm{T}}\right]\\
&= \left[\exp\left(\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}s^{\mu\nu}\right)\right]^{\mathrm{T}} = d^{-1\mathrm{T}}(\Lambda)
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- 这里 $d^{-1\mathrm{T}}(\Lambda) = [d^{-1\dagger}(\Lambda)]^*$，线性表示 $\{d^{-1\mathrm{T}}(\Lambda)\}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ 的**复共轭表示**
- 将 Pauli 矩阵 σ^2 看作一个**幺正变换矩阵**，满足 $(\sigma^2)^{-1} = (\sigma^2)^\dagger = \sigma^2$
- 则 $d(\Lambda)$ 与 $d^{-1\mathrm{T}}(\Lambda)$ 由一个**相似变换**联系起来，相似变换矩阵为 σ^2
- 根据 1.4 节定义，线性表示 $\{d(\Lambda)\}$ 和 $\{d^{-1\mathrm{T}}(\Lambda)\}$ 是**等价**的
- 由于 $(\sigma^2)^* = -\sigma^2$，$\sigma^2 d(\Lambda)\sigma^2 = d^{-1\mathrm{T}}(\Lambda)$ 的复共轭为 $\sigma^2 d^*(\Lambda)\sigma^2 = d^{-1\dagger}(\Lambda)$
- 可见，线性表示 $\{d(\Lambda)\}$ 的**复共轭表示** $\{d^*(\Lambda)\}$ 与 $\{d^{-1\dagger}(\Lambda)\}$ **等价**

左手 Weyl 旋量

于是，**左手 Weyl 旋量**

$$\eta_a = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

的固有保时向 Lorentz 变换为

$$\eta'_a = [d(\Lambda)]_a{}^b \eta_b, \quad a, b = 1, 2$$

η_a 是 $\{d(\Lambda)\}$ **表示空间**中的**列矢量**

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引入**反对称**的**二维 Levi-Civita 符号** ε^{ab}，定义为

$$\varepsilon^{12} = -\varepsilon^{21} = 1, \quad \varepsilon^{11} = \varepsilon^{22} = 0$$

它与 Pauli 矩阵 σ^2 的关系是

$$\varepsilon^{ab} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \mathrm{i}\begin{pmatrix} & -\mathrm{i} \\ \mathrm{i} & \end{pmatrix} = (\mathrm{i}\sigma^2)^{ab}$$

等价的左手 Weyl 旋量

通过 ε^{ab} 定义

$$\eta^a \equiv \varepsilon^{ab}\eta_b = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ -\eta_1 \end{pmatrix}$$

则

$$\eta^1 = \eta_2, \quad \eta^2 = -\eta_1$$

$\sigma^2 d(\Lambda)\sigma^2 = d^{-1\mathrm{T}}(\Lambda)$ 等价于 $\sigma^2 d(\Lambda) = d^{-1\mathrm{T}}(\Lambda)\sigma^2$，故 η^a 的 Lorentz 变换为

$$\begin{aligned} \eta'^a &= \varepsilon^{ab}\eta'_b = \varepsilon^{ab}[d(\Lambda)]_b{}^c\eta_c = \mathrm{i}[\sigma^2 d(\Lambda)]^{ac}\eta_c \\ &= \mathrm{i}[d^{-1\mathrm{T}}(\Lambda)\sigma^2]^{ac}\eta_c = [d^{-1\mathrm{T}}(\Lambda)]^a{}_b\varepsilon^{bc}\eta_c \end{aligned}$$

等价的左手 Weyl 旋量

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- 即

$$\boxed{\eta'^a = [d^{-1\mathrm{T}}(\Lambda)]^a{}_b\eta^b}$$

- 可见 η^a 是 $\{d^{-1\mathrm{T}}(\Lambda)\}$ **表示空间**中的**列矢量**
- 由于 $\{d^{-1\mathrm{T}}(\Lambda)\}$ **等价**于 $\{d(\Lambda)\}$，η^a 也是**左手 Weyl 旋量**

ε^{ab} 和 ε_{ab}

- 两种左手 Weyl 旋量 η_a 与 η^a 是**等价**的，它们之间的关系类似于 Lorentz 逆变矢量 A^μ 与协变矢量 $A_\mu = g_{\mu\nu}A^\nu$ 之间的关系
- ε^{ab} 的作用类似于度规 $g_{\mu\nu}$，相当于 **2 维旋量空间的“度规”**，用于**提升**旋量指标

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- ε^{ab} 的作用类似于度规 $g_{\mu\nu}$，相当于 **2 维旋量空间的“度规”**，用于**提升**旋量指标
- 用 $\varepsilon_{12} = -\varepsilon_{21} = -1$ 和 $\varepsilon_{11} = \varepsilon_{22} = 0$ 定义 ε_{ab}，则

$$\varepsilon_{ab} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -\mathrm{i}\begin{pmatrix} & -\mathrm{i} \\ \mathrm{i} & \end{pmatrix} = (-\mathrm{i}\sigma^2)_{ab}$$

- ε_{ab} 是 ε^{ab} 的**逆矩阵**，满足

$$\varepsilon_{ab}\varepsilon^{bc} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \delta_a{}^c$$

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- 于是，$\eta^1 = \eta_2$ 和 $\eta^2 = -\eta_1$ 表明

$$\eta_a = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta^2 \\ \eta^1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} = \varepsilon_{ab}\eta^b$$

- 也就是说，ε_{ab} 用于**下降**旋量指标

左手 Weyl 旋量的内积

任意两个**左手 Weyl 旋量** η_a 和 ζ_a 的**内积**

$$\eta^a\zeta_a = \varepsilon^{ab}\eta_b\zeta_a = \varepsilon_{ab}\eta^a\zeta^b$$

在固有保时向 Lorentz 变换下**不变**，满足

$$\eta'^a\zeta'_a = [d^{-1\mathrm{T}}(\Lambda)]^a{}_b\eta^b[d(\Lambda)]_a{}^c\zeta_c = \eta^b[d^{-1}(\Lambda)]_b{}^a[d(\Lambda)]_a{}^c\zeta_c = \eta^b\delta_b{}^c\zeta_c = \eta^a\zeta_a$$

第二步用了**转置**性质 $[d^{-1\mathrm{T}}(\Lambda)]^a{}_b = [d^{-1}(\Lambda)]_b{}^a$，可见 $\eta^a\zeta_a$ 是 **Lorentz 标量**

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- 第二步用了**转置**性质 $[d^{-1\mathrm{T}}(\Lambda)]^a{}_b = [d^{-1}(\Lambda)]_b{}^a$，可见 $\eta^a\zeta_a$ 是 **Lorentz 标量**
- 由 $\eta^1 = \eta_2$、$\eta^2 = -\eta_1$、$\zeta^1 = \zeta_2$ 和 $\zeta^2 = -\zeta_1$ 得

$$\eta^a\zeta_a = \eta^1\zeta_1 + \eta^2\zeta_2 = \eta_2\zeta_1 - \eta_1\zeta_2 = -\eta_2\zeta^2 - \eta_1\zeta^1 = -\eta_a\zeta^a$$

- 即参与缩并的**旋量指标一升一降**会多出一个**负号**
- 这种性质与 Lorentz 矢量内积 $A^\mu B_\mu = A_\mu B^\mu$ **截然不同**
- 原因在于旋量空间度规 ε^{ab} 是**反对称**的

Grassmann 数

- $\eta^a\zeta_a = -\eta_a\zeta^a$ 表明 $\boxed{\eta^a\eta_a = -\eta_a\eta^a}$，若 η_a 和 η^a 是**普通的复数**，则 $\eta^a\eta_a = 0$
- 为了使 $\eta^a\eta_a \neq 0$，必须要求左手 Weyl 旋量 η^a 与 η_a **反对易**
- 即它们是 **Grassmann 数**，**任意两个 Grassmann 数都是反对易的**
- 以复数作为组合系数，则若干个 Grassmann 数的**线性组合**也是 Grassmann 数
- 因此，η_a 是 Grassmann 数意味着 $\eta^a = \varepsilon^{ab}\eta_b$ 也是 Grassmann 数

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- 因此，η_a 是 Grassmann 数意味着 $\eta^a = \varepsilon^{ab}\eta_b$ 也是 Grassmann 数
- 虽然如此，Grassmann 数是**反对易的 c 数**，不是算符
- 对 Grassmann 数表达的旋量场进行**量子化**，才得到**旋量场算符**，而 Grassmann 数的反对易性质与旋量场算符的反对易关系**相匹配**

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- 假设 η_a 和 ζ^a 都是 Grassmann 数，则 $\eta_a\zeta^a = -\zeta^a\eta_a$，相应地，将**省略旋量指标**的**内积**写成 $\boxed{\eta\zeta \equiv \eta^a\zeta_a = -\eta_a\zeta^a = \zeta^a\eta_a = \zeta\eta}$，即内积 $\eta\zeta$ 和 $\zeta\eta$ 是**相等**的
- 内积 $\eta^a\eta_a$ 有**等价**表达式

$$\eta\eta = \eta^a\eta_a = \varepsilon_{ab}\eta^a\eta^b = -\eta^1\eta^2 + \eta^2\eta^1 = -2\eta^1\eta^2$$
$$= 2\eta_2\eta_1 = \eta_2\eta_1 - \eta_1\eta_2 = -\varepsilon^{ab}\eta_a\eta_b = -\eta_a\eta^a$$

左手 Weyl 旋量的复共轭

- 将左手 Weyl 旋量 η_a 的复共轭记为 $\eta^\dagger_{\dot{a}} = \begin{pmatrix} \eta^\dagger_{\dot{1}} \\ \eta^\dagger_{\dot{2}} \end{pmatrix}$
- 量子化之后，算符 η_a 和 $\eta^\dagger_{\dot{a}}$ 互为厄米共轭
- 对 $\eta'_a = [d(\Lambda)]_a{}^b \eta_b$ 两边取复共轭，得到 $\eta^\dagger_{\dot{a}}$ 的 Lorentz 变换

$$\eta'^\dagger_{\dot{a}} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} \eta^\dagger_{\dot{b}}$$

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$$\eta'^\dagger_{\dot a} = [d^*(\Lambda)]_{\dot a}{}^{\dot b} \eta^\dagger_{\dot b}$$

- 引进**指标**上带着**点号**的**二维 Levi-Civita 符号**

$$\varepsilon^{\dot a \dot b} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = (\mathrm{i}\sigma^2)^{\dot a \dot b}, \quad \varepsilon_{\dot a \dot b} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = (-\mathrm{i}\sigma^2)_{\dot a \dot b}$$

- 其分量数值与 ε^{ab} 和 ε_{ab} 分别**相同**
- 定义 $\eta^{\dagger \dot a} \equiv \varepsilon^{\dot a \dot b} \eta^\dagger_{\dot b} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \eta^\dagger_{\dot 1} \\ \eta^\dagger_{\dot 2} \end{pmatrix} = \begin{pmatrix} \eta^\dagger_{\dot 2} \\ -\eta^\dagger_{\dot 1} \end{pmatrix}$，则有 $\eta^{\dagger \dot 1} = \eta^\dagger_{\dot 2}$ 和 $\eta^{\dagger \dot 2} = -\eta^\dagger_{\dot 1}$

右手 Weyl 旋量

- $\sigma^2 d^*(\Lambda)\sigma^2 = d^{-1\dagger}(\Lambda)$ 等价于 $\sigma^2 d^*(\Lambda) = d^{-1\dagger}(\Lambda)\sigma^2$
- 故 $\eta^{\dagger\dot{a}}$ 的 Lorentz 变换为

$$\begin{aligned}\eta'^{\dagger\dot{a}} &= \varepsilon^{\dot{a}\dot{b}}\eta'^{\dagger}_{\dot{b}} = \varepsilon^{\dot{a}\dot{b}}[d^*(\Lambda)]_{\dot{b}}{}^{\dot{c}}\eta^{\dagger}_{\dot{c}} = \mathrm{i}[\sigma^2 d^*(\Lambda)]^{\dot{a}\dot{c}}\eta^{\dagger}_{\dot{c}} \\ &= \mathrm{i}[d^{-1\dagger}(\Lambda)\sigma^2]^{\dot{a}\dot{c}}\eta^{\dagger}_{\dot{c}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\varepsilon^{\dot{b}\dot{c}}\eta^{\dagger}_{\dot{c}}\end{aligned}$$

- 即

$$\eta'^{\dagger\dot{a}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\eta^{\dagger\dot{b}}$$

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$$\begin{aligned}\eta'^{\dagger\dot{a}} &= \varepsilon^{\dot{a}\dot{b}}\eta'^{\dagger}_{\dot{b}} = \varepsilon^{\dot{a}\dot{b}}[d^*(\Lambda)]_{\dot{b}}{}^{\dot{c}}\eta^{\dagger}_{\dot{c}} = \mathrm{i}[\sigma^2 d^*(\Lambda)]^{\dot{a}\dot{c}}\eta^{\dagger}_{\dot{c}} \\ &= \mathrm{i}[d^{-1\dagger}(\Lambda)\sigma^2]^{\dot{a}\dot{c}}\eta^{\dagger}_{\dot{c}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\varepsilon^{\dot{b}\dot{c}}\eta^{\dagger}_{\dot{c}}\end{aligned}$$

- 即

$$\eta'^{\dagger\dot{a}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\eta^{\dagger\dot{b}}$$

- 可见，$\eta^{\dagger\dot{a}}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ **表示空间**中的**列矢量**，因而是**右手 Weyl 旋量**
- 由于表示 $\{d^*(\Lambda)\}$ **等价**于 $\{d^{-1\dagger}(\Lambda)\}$，$\eta^{\dagger}_{\dot{a}}$ 也是**右手 Weyl 旋量**
- 因此，在这套符号约定中，**不带点**的旋量指标对应于**左手 Weyl 旋量**及其**表示**
- 而**带点**的旋量指标对应于**右手 Weyl 旋量**及其**表示**

右手 Weyl 旋量的内积

任意两个**右手 Weyl 旋量** $\eta^{\dagger\dot{a}}$ 和 $\zeta^{\dagger\dot{a}}$ 的**内积**

$$\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = \varepsilon_{\dot{a}\dot{b}}\eta^{\dagger\dot{b}}\zeta^{\dagger\dot{a}} = \varepsilon^{\dot{a}\dot{b}}\eta^\dagger_{\dot{a}}\zeta^\dagger_{\dot{b}}$$

在固有保时向 Lorentz 变换下**不变**，满足

$$\eta'^\dagger_{\dot{a}}\zeta'^{\dagger\dot{a}} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}}\eta^\dagger_{\dot{b}}[d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{b}}[d^\dagger(\Lambda)]^{\dot{b}}{}_{\dot{a}}[d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{b}}\delta^{\dot{b}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}}$$

第二步用了**转置**性质 $[d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} = [d^\dagger(\Lambda)]^{\dot{b}}{}_{\dot{a}}$，可见 $\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}}$ 是 **Lorentz 标量**

右手 Weyl 旋量的内积

任意两个**右手 Weyl 旋量** $\eta^{\dagger\dot a}$ 和 $\zeta^{\dagger\dot a}$ 的**内积**

$$\eta^\dagger_{\dot a}\zeta^{\dagger\dot a} = \varepsilon_{\dot a\dot b}\eta^{\dagger\dot b}\zeta^{\dagger\dot a} = \varepsilon^{\dot a\dot b}\eta^\dagger_{\dot a}\zeta^\dagger_{\dot b}$$

在固有保时向 Lorentz 变换下**不变**，满足

$$\eta'^\dagger_{\dot a}\zeta'^{\dagger\dot a} = [d^*(\Lambda)]_{\dot a}{}^{\dot b}\eta^\dagger_{\dot b}[d^{-1\dagger}(\Lambda)]^{\dot a}{}_{\dot c}\zeta^{\dagger\dot c} = \eta^\dagger_{\dot b}[d^\dagger(\Lambda)]^{\dot b}{}_{\dot a}[d^{-1\dagger}(\Lambda)]^{\dot a}{}_{\dot c}\zeta^{\dagger\dot c} = \eta^\dagger_{\dot b}\delta^{\dot b}{}_{\dot c}\zeta^{\dagger\dot c} = \eta^\dagger_{\dot a}\zeta^{\dagger\dot a}$$

第二步用了**转置**性质 $[d^*(\Lambda)]_{\dot a}{}^{\dot b} = [d^\dagger(\Lambda)]^{\dot b}{}_{\dot a}$，可见 $\eta^\dagger_{\dot a}\zeta^{\dagger\dot a}$ 是 **Lorentz 标量**

由 $\eta^{\dagger\dot 1} = \eta^\dagger_{\dot 2}$、$\eta^{\dagger\dot 2} = -\eta^\dagger_{\dot 1}$、$\zeta^{\dagger\dot 1} = \zeta^\dagger_{\dot 2}$ 和 $\zeta^{\dagger\dot 2} = -\zeta^\dagger_{\dot 1}$ 得

$$\eta^\dagger_{\dot a}\zeta^{\dagger\dot a} = \eta^\dagger_{\dot 1}\zeta^{\dagger\dot 1} + \eta^\dagger_{\dot 2}\zeta^{\dagger\dot 2} = -\eta^{\dagger\dot 2}\zeta^{\dagger\dot 1} + \eta^{\dagger\dot 1}\zeta^{\dagger\dot 2} = -\eta^{\dagger\dot 2}\zeta^\dagger_{\dot 2} - \eta^{\dagger\dot 1}\zeta^\dagger_{\dot 1} = -\eta^{\dagger\dot a}\zeta^\dagger_{\dot a}$$

即参与缩并的**带点旋量指标一升一降**会多出一个**负号**

右手 Weyl 旋量的内积

任意两个**右手 Weyl 旋量** $\eta^{\dagger\dot{a}}$ 和 $\zeta^{\dagger\dot{a}}$ 的**内积**

$$\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = \varepsilon_{\dot{a}\dot{b}}\eta^{\dagger\dot{b}}\zeta^{\dagger\dot{a}} = \varepsilon^{\dot{a}\dot{b}}\eta^\dagger_{\dot{a}}\zeta^\dagger_{\dot{b}}$$

在固有保时向 Lorentz 变换下**不变**，满足

$$\eta'^\dagger_{\dot{a}}\zeta'^{\dagger\dot{a}} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}}\eta^\dagger_{\dot{b}}[d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{b}}[d^\dagger(\Lambda)]^{\dot{b}}{}_{\dot{a}}[d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{b}}\delta^{\dot{b}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}}$$

- 第二步用了**转置**性质 $[d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} = [d^\dagger(\Lambda)]^{\dot{b}}{}_{\dot{a}}$，可见 $\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}}$ 是 **Lorentz 标量**
- 由 $\eta^{\dagger\dot{1}} = \eta^\dagger_{\dot{2}}$、$\eta^{\dagger\dot{2}} = -\eta^\dagger_{\dot{1}}$、$\zeta^{\dagger\dot{1}} = \zeta^\dagger_{\dot{2}}$ 和 $\zeta^{\dagger\dot{2}} = -\zeta^\dagger_{\dot{1}}$ 得

$$\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = \eta^\dagger_{\dot{1}}\zeta^{\dagger\dot{1}} + \eta^\dagger_{\dot{2}}\zeta^{\dagger\dot{2}} = -\eta^{\dagger\dot{2}}\zeta^{\dagger\dot{1}} + \eta^{\dagger\dot{1}}\zeta^{\dagger\dot{2}} = -\eta^{\dagger\dot{2}}\zeta^\dagger_{\dot{2}} - \eta^{\dagger\dot{1}}\zeta^\dagger_{\dot{1}} = -\eta^{\dagger\dot{a}}\zeta^\dagger_{\dot{a}}$$

- 即参与缩并的**带点旋量指标一升一降**会多出一个**负号**
- 假设右手 Weyl 旋量 $\eta^{\dagger\dot{a}}$ 和 $\zeta^\dagger_{\dot{a}}$ 都是 **Grassmann 数**，则 $\eta^{\dagger\dot{a}}\zeta^\dagger_{\dot{a}} = -\zeta^\dagger_{\dot{a}}\eta^{\dagger\dot{a}}$
- 将**省略**带点旋量指标的**内积**写成 $\eta^\dagger\zeta^\dagger \equiv \eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = -\eta^{\dagger\dot{a}}\zeta^\dagger_{\dot{a}} = \zeta^\dagger_{\dot{a}}\eta^{\dagger\dot{a}} = \zeta^\dagger\eta^\dagger$
- 则内积 $\eta^\dagger\zeta^\dagger$ 和 $\zeta^\dagger\eta^\dagger$ **相等**

Lorentz 不变量和 Weyl 旋量算符

🐂 可以看到，只要将不带点和带点的旋量指标分别缩并完毕，就得到 **Lorentz 标量**

🎩 另一方面，缩并一个不带点的指标和一个带点的指标并不能得到 Lorentz 不变量

👓 比如，$\eta^a \zeta^\dagger_{\dot{a}}$ 和 $\eta^{\dagger\dot{a}}\zeta_a$ 都**不是 Lorentz 标量**

Lorentz 不变量和 Weyl 旋量算符

- 可以看到，只要将不带点和带点的旋量指标分别缩并完毕，就得到 **Lorentz 标量**
- 另一方面，缩并一个不带点的指标和一个带点的指标并不能得到 Lorentz 不变量
- 比如，$\eta^a \zeta^\dagger_{\dot{a}}$ 和 $\eta^{\dagger\dot{a}}\zeta_a$ 都不是 **Lorentz 标量**
- 对于 **Weyl 旋量算符** η_a 和 ζ_a，有

$$(\eta\zeta)^\dagger = (\eta^a\zeta_a)^\dagger = (\zeta_a)^\dagger(\eta^a)^\dagger = \zeta^\dagger_{\dot{a}}\eta^{\dagger\dot{a}} = \zeta^\dagger\eta^\dagger$$

- 即 $\zeta^\dagger\eta^\dagger$ 是 $\eta\zeta$ 的厄米共轭算符
- 厄米共轭操作将左手和右手 Weyl 旋量算符相互转换

9.6.2 小节 Dirac 和 Majorana 旋量场的分解

- 依照上一小节关于旋量指标的约定，将 **Dirac 旋量场** $\psi(x)$ 分解成**左手 Weyl 旋量场** $\eta_a(x)$ 和**右手 Weyl 旋量场** $\zeta^{\dagger\dot{a}}(x)$ ，形式为

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \zeta^{\dagger\dot{a}}(x) \end{pmatrix}$$

- 在量子化之前，$\eta_a(x)$ 和 $\zeta^{\dagger\dot{a}}(x)$ 是 Grassmann 数，因而 $\psi(x)$ 也是 **Grassmann 数**
- 这是在 9.2.1 小节中**转置两个旋量场**必须添加一个额外**负号**的原因

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这是在 9.2.1 小节中**转置两个旋量场**必须添加一个额外**负号**的原因

根据 $D(\Lambda) = \begin{pmatrix} d(\Lambda) & \\ & d^{-1\dagger}(\Lambda) \end{pmatrix}$，$\psi(x)$ 的**固有保时向 Lorentz 变换**表达成

$$\begin{pmatrix} \eta'_a(x') \\ \zeta'^{\dagger\dot{a}}(x') \end{pmatrix} = \psi'(x') = D(\Lambda)\psi(x) = \begin{pmatrix} [d(\Lambda)]_a{}^b\eta_b(x) \\ [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\zeta^{\dagger\dot{b}}(x) \end{pmatrix}$$

$\psi(x)$ 的 **Dirac 共轭**是 $\bar{\psi} = \psi^\dagger\gamma^0 = \begin{pmatrix} \eta^\dagger_{\dot{b}} & \zeta^b \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}{}_{\dot{a}} \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & \eta^\dagger_{\dot{a}} \end{pmatrix}$

Dirac 矩阵的指标形式

保持**旋量指标平衡**，则 **Dirac 方程** $(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0$ 化为

$$\begin{pmatrix} -m\delta_a{}^b & \mathrm{i}(\sigma^\mu)_{a\dot{b}}\partial_\mu \\ \mathrm{i}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu & -m\delta^{\dot{a}}{}_{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dagger\dot{b}} \end{pmatrix} = 0$$

因而 **Dirac 矩阵**的指标形式是

$$\gamma^\mu = \begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}$$

注意，γ^μ 中的 γ^0 与 Dirac 共轭 $\bar{\psi} = \psi^\dagger\gamma^0 = \begin{pmatrix} \eta^\dagger_{\dot{a}} & \zeta^a \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}{}_{\dot{a}} \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & \eta^\dagger_{\dot{a}} \end{pmatrix}$ 中的 γ^0 具有**不同的指标结构**

两者本质不同，有些书将后者记为 β 以示区别

σ^μ 和 $\bar{\sigma}^\mu$ 的 Lorentz 变换规则

于是，γ^μ 的 **Lorentz 变换规则** $D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$ 左边变成

$$\begin{aligned} & D^{-1}(\Lambda)\gamma^\mu D(\Lambda) \\ = & \begin{pmatrix} [d^{-1}(\Lambda)]_a{}^c & \\ & [d^\dagger(\Lambda)]^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{c\dot{d}} \\ (\bar{\sigma}^\mu)^{\dot{c}d} & \end{pmatrix} \begin{pmatrix} [d(\Lambda)]_d{}^b & \\ & [d^{-1\dagger}(\Lambda)]^{\dot{d}}{}_{\dot{b}} \end{pmatrix} \\ = & \begin{pmatrix} & [d^{-1}(\Lambda)]_a{}^c(\sigma^\mu)_{c\dot{d}}[d^{-1\dagger}(\Lambda)]^{\dot{d}}{}_{\dot{b}} \\ [d^\dagger(\Lambda)]^{\dot{a}}{}_{\dot{c}}(\bar{\sigma}^\mu)^{\dot{c}d}[d(\Lambda)]_d{}^b & \end{pmatrix} \end{aligned}$$

右边化为

$$\Lambda^\mu{}_\nu\gamma^\nu = \begin{pmatrix} & \Lambda^\mu{}_\nu(\sigma^\nu)_{a\dot{b}} \\ \Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot{a}b} & \end{pmatrix}$$

两相比较，推出

$$[d^{-1}(\Lambda)]_a{}^c(\sigma^\mu)_{c\dot{d}}[d^{-1\dagger}(\Lambda)]^{\dot{d}}{}_{\dot{b}} = \Lambda^\mu{}_\nu(\sigma^\nu)_{a\dot{b}}, \quad [d^\dagger(\Lambda)]^{\dot{a}}{}_{\dot{c}}(\bar{\sigma}^\mu)^{\dot{c}d}[d(\Lambda)]_d{}^b = \Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot{a}b}$$

这分别是 σ^μ 和 $\bar{\sigma}^\mu$ 的 **Lorentz 变换规则**

Lorentz 矢量 $\eta\sigma^\mu\zeta^\dagger$ 和 $\eta^\dagger\bar{\sigma}^\mu\zeta$

🐥 对任意 **Weyl 旋量** η 和 ζ，定义 $\eta\sigma^\mu\zeta^\dagger \equiv \eta^a(\sigma^\mu)_{a\dot{b}}\zeta^{\dagger\dot{b}}$, $\quad \eta^\dagger\bar{\sigma}^\mu\zeta \equiv \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\zeta_b$

🧧 它们都是 **Lorentz 矢量**，相应的固有保时向 Lorentz 变换为

$$\begin{aligned}
\eta'\sigma^\mu\zeta'^\dagger &= [d^{-1\mathrm{T}}(\Lambda)]^a{}_c\eta^c(\sigma^\mu)_{a\dot{b}}[d^{-1\dagger}(\Lambda)]^{\dot{b}}{}_{\dot{d}}\zeta^{\dagger\dot{d}} = \eta^c[d^{-1}(\Lambda)]_c{}^a(\sigma^\mu)_{a\dot{b}}[d^{-1\dagger}(\Lambda)]^{\dot{b}}{}_{\dot{d}}\zeta^{\dagger\dot{d}} \\
&= \eta^c\Lambda^\mu{}_\nu(\sigma^\nu)_{c\dot{d}}\zeta^{\dagger\dot{d}} = \Lambda^\mu{}_\nu\eta\sigma^\nu\zeta^\dagger \\
\eta'^\dagger\bar{\sigma}^\mu\zeta' &= [d^*(\Lambda)]_{\dot{a}}{}^{\dot{c}}\eta^\dagger_{\dot{c}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b{}^d\zeta_d = \eta^\dagger_{\dot{c}}[d^\dagger(\Lambda)]^{\dot{c}}{}_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b{}^d\zeta_d \\
&= \eta^\dagger_{\dot{c}}\Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot{c}d}\zeta_d = \Lambda^\mu{}_\nu\eta^\dagger\bar{\sigma}^\mu\zeta
\end{aligned}$$

Lorentz 矢量 $\eta\sigma^\mu\zeta^\dagger$ 和 $\eta^\dagger\bar\sigma^\mu\zeta$

对任意 **Weyl 旋量** η 和 ζ，定义 $\eta\sigma^\mu\zeta^\dagger \equiv \eta^a(\sigma^\mu)_{a\dot b}\zeta^{\dagger\dot b}$, $\quad \eta^\dagger\bar\sigma^\mu\zeta \equiv \eta^\dagger_{\dot a}(\bar\sigma^\mu)^{\dot a b}\zeta_b$

它们都是 **Lorentz 矢量**，相应的固有保时向 Lorentz 变换为

$$\eta'\sigma^\mu\zeta'^\dagger = [d^{-1\mathrm{T}}(\Lambda)]^a{}_c\eta^c(\sigma^\mu)_{a\dot b}[d^{-1\dagger}(\Lambda)]^{\dot b}{}_{\dot d}\zeta^{\dagger\dot d} = \eta^c[d^{-1}(\Lambda)]_c{}^a(\sigma^\mu)_{a\dot b}[d^{-1\dagger}(\Lambda)]^{\dot b}{}_{\dot d}\zeta^{\dagger\dot d}$$
$$= \eta^c\Lambda^\mu{}_\nu(\sigma^\nu)_{c\dot d}\zeta^{\dagger\dot d} = \Lambda^\mu{}_\nu\eta\sigma^\nu\zeta^\dagger$$

$$\eta'^\dagger\bar\sigma^\mu\zeta' = [d^*(\Lambda)]_{\dot a}{}^{\dot c}\eta^\dagger_{\dot c}(\bar\sigma^\mu)^{\dot a b}[d(\Lambda)]_b{}^d\zeta_d = \eta^\dagger_{\dot c}[d^\dagger(\Lambda)]^{\dot c}{}_{\dot a}(\bar\sigma^\mu)^{\dot a b}[d(\Lambda)]_b{}^d\zeta_d$$
$$= \eta^\dagger_{\dot c}\Lambda^\mu{}_\nu(\bar\sigma^\nu)^{\dot c d}\zeta_d = \Lambda^\mu{}_\nu\eta^\dagger\bar\sigma^\mu\zeta$$

由 $\sigma^2\sigma^\mu\sigma^2 = (\bar\sigma^\mu)^\mathrm{T}$ 得 $(\mathrm{i}\sigma^2)\sigma^\mu(\mathrm{i}\sigma^2) = -(\bar\sigma^\mu)^\mathrm{T}$，相应的指标形式为

$$\varepsilon^{ac}(\sigma^\mu)_{c\dot d}\varepsilon^{\dot d\dot b} = -[(\bar\sigma^\mu)^\mathrm{T}]^{a\dot b} = -(\bar\sigma^\mu)^{\dot b a}$$

对于 **Weyl 旋量场** $\eta_a(x)$ 和 $\zeta^{\dagger\dot a}(x)$，有 Grassmann 数性质

$$[\eta^a(\sigma^\mu)_{a\dot b}\zeta^{\dagger\dot b}]^\dagger = \zeta^b(\sigma^\mu)_{b\dot a}\eta^{\dagger\dot a} = -\eta^{\dagger\dot a}(\sigma^\mu)_{b\dot a}\zeta^b = -\varepsilon^{\dot a\dot c}\eta^\dagger_{\dot c}(\sigma^\mu)_{b\dot a}\varepsilon^{bd}\zeta_d$$
$$= \eta^\dagger_{\dot c}\varepsilon^{db}(\sigma^\mu)_{b\dot a}\varepsilon^{\dot a\dot c}\zeta_d = -\eta^\dagger_{\dot c}(\bar\sigma^\mu)^{\dot c d}\zeta_d = -[\zeta^\dagger_{\dot d}(\bar\sigma^\mu)^{\dot d c}\eta_c]^\dagger$$

即 $(\eta\sigma^\mu\zeta^\dagger)^\dagger = \zeta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar\sigma^\mu\zeta = -(\zeta^\dagger\bar\sigma^\mu\eta)^\dagger$

Lorentz 张量 $\eta\sigma^\mu\bar{\sigma}^\nu\zeta$ 和 $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger$

- 类似地，$\eta\sigma^\mu\bar{\sigma}^\nu\zeta \equiv \eta^a(\sigma^\mu)_{a\dot{b}}(\bar{\sigma}^\nu)^{\dot{b}c}\zeta_c$ 和 $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger \equiv \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}(\sigma^\nu)_{b\dot{c}}\zeta^{\dagger\dot{c}}$ 都是**二阶 Lorentz 张量**

- 由 $\sigma^2\bar{\sigma}^\mu\sigma^2 = (\sigma^\mu)^{\mathrm{T}}$ 得 $(-\mathrm{i}\sigma^2)\bar{\sigma}^\mu(-\mathrm{i}\sigma^2) = -(\sigma^\mu)^{\mathrm{T}}$，相应的指标形式为

$$\varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\mu)^{\dot{c}d}\varepsilon_{db} = -[(\sigma^\mu)^{\mathrm{T}}]_{\dot{a}b} = -(\sigma^\mu)_{b\dot{a}}$$

- 再利用 $\varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c$ 和 $\varepsilon^{ac}(\sigma^\mu)_{c\dot{d}}\varepsilon^{\dot{d}\dot{b}} = -[(\bar{\sigma}^\mu)^{\mathrm{T}}]^{a\dot{b}} = -(\bar{\sigma}^\mu)^{\dot{b}a}$ 推出

$$\begin{aligned}\varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}(\sigma^\mu)_{d\dot{e}}\varepsilon^{\dot{e}\dot{b}} &= \varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}\delta_d{}^f(\sigma^\mu)_{f\dot{e}}\varepsilon^{\dot{e}\dot{b}} = \varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}\varepsilon_{dg}\varepsilon^{gf}(\sigma^\mu)_{f\dot{e}}\varepsilon^{\dot{e}\dot{b}}\\ &= (-\sigma^\nu)_{g\dot{a}}(-\bar{\sigma}^\mu)^{\dot{b}g} = (\bar{\sigma}^\mu)^{\dot{b}g}(\sigma^\nu)_{g\dot{a}}\end{aligned}$$

- 故

$$\begin{aligned}[\eta^a(\sigma^\mu)_{a\dot{b}}(\bar{\sigma}^\nu)^{\dot{b}c}\zeta_c]^\dagger &= \zeta^\dagger_{\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\eta^{\dagger\dot{a}} = -\eta^{\dagger\dot{a}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\zeta^\dagger_{\dot{c}}\\ &= -\varepsilon^{\dot{a}\dot{d}}\eta^\dagger_{\dot{d}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\varepsilon_{\dot{c}\dot{e}}\zeta^{\dagger\dot{e}} = \eta^\dagger_{\dot{d}}\varepsilon_{\dot{e}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\varepsilon^{\dot{a}\dot{d}}\zeta^{\dagger\dot{e}}\\ &= \eta^\dagger_{\dot{d}}(\bar{\sigma}^\mu)^{\dot{d}g}(\sigma^\nu)_{g\dot{e}}\zeta^{\dagger\dot{e}} = [\zeta^e(\sigma^\nu)_{e\dot{g}}(\bar{\sigma}^\mu)^{\dot{g}d}\eta_d]^\dagger\end{aligned}$$

- 即

$$(\eta\sigma^\mu\bar{\sigma}^\nu\zeta)^\dagger = \zeta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger = \eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger = (\zeta\sigma^\nu\bar{\sigma}^\mu\eta)^\dagger$$

旋量双线性型的分解

将 **Dirac 旋量双线性型**分解成由 **Weyl 旋量**表达的 **Lorentz 张量**，有

$$\bar{\psi}\psi = \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}\eta_a \\ \zeta^{\dagger\dot{a}}\end{pmatrix} = \zeta^a\eta_a + \eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = \zeta\eta + \eta^\dagger\zeta^\dagger$$

$$\bar{\psi}\gamma^5\psi = \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}-\delta_a{}^b & \\ & \delta^{\dot{a}}{}_{\dot{b}}\end{pmatrix}\begin{pmatrix}\eta_b \\ \zeta^{\dagger\dot{b}}\end{pmatrix} = -\zeta^a\eta_a + \eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = -\zeta\eta + \eta^\dagger\zeta^\dagger$$

$$\begin{aligned}\bar{\psi}\gamma^\mu\psi &= \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}\begin{pmatrix}\eta_b \\ \zeta^{\dagger\dot{b}}\end{pmatrix} = \zeta^a(\sigma^\mu)_{a\dot{b}}\zeta^{\dagger\dot{b}} + \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\eta_b \\ &= \zeta\sigma^\mu\zeta^\dagger + \eta^\dagger\bar{\sigma}^\mu\eta\end{aligned}$$

$$\begin{aligned}\bar{\psi}\gamma^\mu\gamma^5\psi &= \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}\begin{pmatrix}-\delta_b{}^c & \\ & \delta^{\dot{b}}{}_{\dot{c}}\end{pmatrix}\begin{pmatrix}\eta_c \\ \zeta^{\dagger\dot{c}}\end{pmatrix} \\ &= \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}\begin{pmatrix}-\eta_b \\ \zeta^{\dagger\dot{b}}\end{pmatrix} = \zeta^a(\sigma^\mu)_{a\dot{b}}\zeta^{\dagger\dot{b}} - \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\eta_b \\ &= \zeta\sigma^\mu\zeta^\dagger - \eta^\dagger\bar{\sigma}^\mu\eta\end{aligned}$$

旋量双线性型的分解

还有

$$\begin{aligned}\bar{\psi}\sigma^{\mu\nu}\psi &= \frac{\mathrm{i}}{2}\begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}(\sigma^\mu\bar{\sigma}^\nu-\sigma^\nu\bar{\sigma}^\mu)_a{}^b & \\ & (\bar{\sigma}^\mu\sigma^\nu-\bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}{}_{\dot{b}}\end{pmatrix}\begin{pmatrix}\eta_b \\ \zeta^{\dagger\dot{b}}\end{pmatrix}\\ &= \frac{\mathrm{i}}{2}\zeta^a(\sigma^\mu\bar{\sigma}^\nu-\sigma^\nu\bar{\sigma}^\mu)_a{}^b\eta_b+\frac{\mathrm{i}}{2}\eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu\sigma^\nu-\bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}{}_{\dot{b}}\zeta^{\dagger\dot{b}}\\ &= \frac{\mathrm{i}}{2}\zeta(\sigma^\mu\bar{\sigma}^\nu-\sigma^\nu\bar{\sigma}^\mu)\eta+\frac{\mathrm{i}}{2}\eta^\dagger(\bar{\sigma}^\mu\sigma^\nu-\bar{\sigma}^\nu\sigma^\mu)\zeta^\dagger\end{aligned}$$

进一步推出

$$\begin{aligned}\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} &= \frac{1}{2}\bar{\psi}(1-\gamma^5)\psi=\zeta\eta\\ \bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} &= \frac{1}{2}\bar{\psi}(1+\gamma^5)\psi=\eta^\dagger\zeta^\dagger\\ \bar{\psi}_{\mathrm{L}}\gamma^\mu\psi_{\mathrm{L}} &= \frac{1}{2}\bar{\psi}(\gamma^\mu-\gamma^\mu\gamma^5)\psi=\eta^\dagger\bar{\sigma}^\mu\eta\\ \bar{\psi}_{\mathrm{L}}\gamma^\mu\psi_{\mathrm{R}} &= \frac{1}{2}\bar{\psi}(\gamma^\mu+\gamma^\mu\gamma^5)\psi=\zeta\sigma^\mu\zeta^\dagger\end{aligned}$$

拉氏量的分解

另一方面，自由 Dirac 旋量场的拉氏量分解为

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}-m\delta_a{}^b & \mathrm{i}(\sigma^\mu)_{a\dot{b}}\partial_\mu \\ \mathrm{i}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu & -m\delta^{\dot{a}}{}_{\dot{b}}\end{pmatrix}\begin{pmatrix}\eta_b \\ \zeta^{\dagger\dot{b}}\end{pmatrix} \\ &= -m\zeta^a\eta_a + \mathrm{i}\zeta^a(\sigma^\mu)_{a\dot{b}}\partial_\mu\zeta^{\dagger\dot{b}} + \mathrm{i}\eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu\eta_b - m\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} \\ &= \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta + \mathrm{i}\zeta\sigma^\mu\partial_\mu\zeta^\dagger - m(\zeta\eta + \eta^\dagger\zeta^\dagger)\end{aligned}$$

这里的质量项涉及两个不同的 Weyl 旋量场 $\eta_a(x)$ 和 $\zeta_a(x)$，称为 **Dirac 质量项**

如果质量 $m = 0$，则

$$\mathcal{L}_\mathrm{L} = \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta \quad \text{和} \quad \mathcal{L}_\mathrm{R} = \mathrm{i}\zeta\sigma^\mu\partial_\mu\zeta^\dagger$$

分别描述自由的左手 Weyl 旋量场 $\eta_a(x)$ 和右手 Weyl 旋量场 $\zeta^{\dagger\dot{a}}(x)$

相应的运动方程是两个 **Weyl 方程**：

$$\mathrm{i}\bar{\sigma}^\mu\partial_\mu\eta = 0, \quad \mathrm{i}\sigma^\mu\partial_\mu\zeta^\dagger = 0$$

Weyl 旋量场的 C 变换

- 下面讨论 **Weyl 旋量场**的**分立变换**
- 首先，**电荷共轭矩阵**的指标形式为 $\mathcal{C} = \begin{pmatrix} -\mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}$
- 将 $\psi(x)$ 的**电荷共轭场** $\psi^{\mathrm{C}}(x)$ 分解成 Weyl 旋量场，得到

$$\psi^{\mathrm{C}}(x) = \mathcal{C}\bar{\psi}^{\mathrm{T}}(x) = \mathcal{C}\begin{pmatrix} \zeta^b(x) & \eta^\dagger_{\dot{b}}(x) \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \zeta^b(x) \\ \eta^\dagger_{\dot{b}}(x) \end{pmatrix} = \begin{pmatrix} \zeta_a(x) \\ \eta^{\dagger\dot{a}}(x) \end{pmatrix}$$

- 从而，**Dirac 旋量场** $\psi(x)$ 的 **C 变换**化为

$$\begin{pmatrix} C^{-1}\eta_a(x)C \\ C^{-1}\zeta^{\dagger\dot{a}}(x)C \end{pmatrix} = C^{-1}\psi(x)C = \zeta_C^*\psi^{\mathrm{C}}(x) = \begin{pmatrix} \zeta_C^*\zeta_a(x) \\ \zeta_C^*\eta^{\dagger\dot{a}}(x) \end{pmatrix}$$

Weyl 旋量场的 C 变换

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- 首先，**电荷共轭矩阵**的指标形式为 $\mathcal{C} = \begin{pmatrix} -\mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}$
- 将 $\psi(x)$ 的**电荷共轭场** $\psi^{\mathrm{C}}(x)$ 分解成 Weyl 旋量场，得到

$$\psi^{\mathrm{C}}(x) = \mathcal{C}\bar{\psi}^{\mathrm{T}}(x) = \mathcal{C}\begin{pmatrix} \zeta^b(x) & \eta^\dagger_{\dot{b}}(x) \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}\begin{pmatrix} \zeta^b(x) \\ \eta^\dagger_{\dot{b}}(x) \end{pmatrix} = \begin{pmatrix} \zeta_a(x) \\ \eta^{\dagger\dot{a}}(x) \end{pmatrix}$$

- 从而，**Dirac 旋量场** $\psi(x)$ 的 **C 变换**化为

$$\begin{pmatrix} C^{-1}\eta_a(x)C \\ C^{-1}\zeta^{\dagger\dot{a}}(x)C \end{pmatrix} = C^{-1}\psi(x)C = \zeta_C^*\psi^{\mathrm{C}}(x) = \begin{pmatrix} \zeta_C^*\zeta_a(x) \\ \zeta_C^*\eta^{\dagger\dot{a}}(x) \end{pmatrix}$$

- 即**左右**手 Weyl 旋量场的 **C 变换**是

$$C^{-1}\eta_a(x)C = \zeta_C^*\zeta_a(x), \quad C^{-1}\zeta^{\dagger\dot{a}}(x)C = \zeta_C^*\eta^{\dagger\dot{a}}(x)$$

- 可见，电荷共轭变换将 **η 和 ζ 相互转换**。取厄米共轭，得 $C^{-1}\eta^\dagger_{\dot{b}}(x)C = \zeta_C\zeta^\dagger_{\dot{b}}(x)$ 及 $C^{-1}\zeta^b(x)C = \zeta_C\eta^b(x)$，分别与 $\varepsilon^{\dot{a}\dot{b}}$ 和 ε_{ab} 缩并，推出

$$C^{-1}\eta^{\dagger\dot{a}}(x)C = \zeta_C\zeta^{\dagger\dot{a}}(x), \quad C^{-1}\zeta_a(x)C = \zeta_C\eta_a(x)$$

Weyl 旋量场的 P 变换

- 其次，**Dirac 旋量场** $\psi(x)$ 的 **P 变换**表达为

$$\begin{pmatrix} P^{-1}\eta_a(x)P \\ P^{-1}\zeta^{\dagger\dot{a}}(x)P \end{pmatrix} = P^{-1}\psi(x)P = \zeta_P^*\gamma^0\psi(\mathcal{P}x)$$

$$= \zeta_P^* \begin{pmatrix} & \delta^{\dot{a}}{}_{\dot{b}} \\ \delta_a{}^b & \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{P}x) \\ \zeta^{\dagger\dot{b}}(\mathcal{P}x) \end{pmatrix} = \begin{pmatrix} \zeta_P^*\zeta^{\dagger\dot{a}}(\mathcal{P}x) \\ \zeta_P^*\eta_a(\mathcal{P}x) \end{pmatrix}$$

- 注意此处 γ^0 的指标结构与 $\bar{\psi} = \psi^\dagger\gamma^0$ 中一样

Weyl 旋量场的 P 变换

🐫 其次，**Dirac 旋量场** $\psi(x)$ 的 $\boldsymbol{P}$ **变换**表达为

$$\begin{pmatrix} P^{-1}\eta_a(x)P \\ P^{-1}\zeta^{\dagger\dot{a}}(x)P \end{pmatrix} = P^{-1}\psi(x)P = \zeta_P^*\gamma^0\psi(\mathcal{P}x)$$

$$= \zeta_P^* \begin{pmatrix} & \delta^{\dot{a}}{}_{\dot{b}} \\ \delta_a{}^b & \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{P}x) \\ \zeta^{\dagger\dot{b}}(\mathcal{P}x) \end{pmatrix} = \begin{pmatrix} \zeta_P^*\zeta^{\dagger\dot{a}}(\mathcal{P}x) \\ \zeta_P^*\eta_a(\mathcal{P}x) \end{pmatrix}$$

🃏 注意此处 γ^0 的指标结构与 $\bar{\psi} = \psi^\dagger\gamma^0$ 中一样

♠ 于是得到**左右**手 Weyl 旋量场的 $\boldsymbol{P}$ **变换**

$$P^{-1}\eta_a(x)P = \zeta_P^*\zeta^{\dagger\dot{a}}(\mathcal{P}x), \quad P^{-1}\zeta^{\dagger\dot{a}}(x)P = \zeta_P^*\eta_a(\mathcal{P}x)$$

♥ 也就是说，宇称变换将**左手**和**右手** Weyl 旋量场**相互转换**

♣ 取**厄米共轭**得 $P^{-1}\eta^\dagger_{\dot{b}}(x)P = \zeta_P\zeta^b(\mathcal{P}x)$ 和 $P^{-1}\zeta^b(x)P = \zeta_P\eta^\dagger_{\dot{b}}(\mathcal{P}x)$

♦ 两边与 $\mathrm{i}\sigma^2 = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{ab}$ 缩并，推出

$$P^{-1}\eta^{\dagger\dot{a}}(x)P = -\zeta_P\zeta_a(\mathcal{P}x), \quad P^{-1}\zeta_a(x)P = -\zeta_P\eta^{\dagger\dot{a}}(\mathcal{P}x)$$

Weyl 旋量场的 T 变换

- 最后，**时间反演矩阵**的指标形式是 $D(\mathcal{T}) = \mathcal{C}\gamma^5 = \begin{pmatrix} \mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon^{ab} & \\ & -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix}$

- **Dirac 旋量场 $\psi(x)$ 的 T 变换**化为

$$\begin{pmatrix} T^{-1}\eta_a(x)T \\ T^{-1}\zeta^{\dagger\dot{a}}(x)T \end{pmatrix} = T^{-1}\psi(x)T = \zeta_T^* \mathcal{C}\gamma^5\psi(\mathcal{T}x)$$

$$= \zeta_T^* \begin{pmatrix} \varepsilon^{ab} & \\ & -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{T}x) \\ \zeta^{\dagger\dot{b}}(\mathcal{T}x) \end{pmatrix} = \begin{pmatrix} \zeta_T^*\eta^a(\mathcal{T}x) \\ -\zeta_T^*\zeta^\dagger_{\dot{a}}(\mathcal{T}x) \end{pmatrix}$$

Weyl 旋量场的 T 变换

🐪 最后，**时间反演矩阵**的指标形式是 $D(\mathcal{T}) = \mathcal{C}\gamma^5 = \begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon^{ab} & \\ & -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix}$

🎴 **Dirac 旋量场 $\psi(x)$ 的 T 变换**化为

$$\begin{pmatrix} T^{-1}\eta_a(x)T \\ T^{-1}\zeta^{\dagger\dot{a}}(x)T \end{pmatrix} = T^{-1}\psi(x)T = \zeta_T^* \mathcal{C}\gamma^5\psi(\mathcal{T}x)$$

$$= \zeta_T^* \begin{pmatrix} \varepsilon^{ab} & \\ & -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{T}x) \\ \zeta^{\dagger\dot{b}}(\mathcal{T}x) \end{pmatrix} = \begin{pmatrix} \zeta_T^*\eta^a(\mathcal{T}x) \\ -\zeta_T^*\zeta^\dagger_{\dot{a}}(\mathcal{T}x) \end{pmatrix}$$

🪄 则**左右**手 Weyl 旋量场的 **T 变换**是

$$T^{-1}\eta_a(x)T = \zeta_T^*\eta^a(\mathcal{T}x), \quad T^{-1}\zeta^{\dagger\dot{a}}(x)T = -\zeta_T^*\zeta^\dagger_{\dot{a}}(\mathcal{T}x)$$

🔮 取**厄米共轭**，有 $T^{-1}\eta^\dagger_{\dot{b}}(x)T = \zeta_T\eta^{\dagger\dot{b}}(\mathcal{T}x)$ 和 $T^{-1}\zeta^b(x)T = -\zeta_T\zeta_b(\mathcal{T}x)$

🪆 与 $i\sigma^2 = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{ab} = \varepsilon^{ab}$ 缩并，得

$$T^{-1}\eta^{\dagger\dot{a}}(x)T = -\zeta_T\eta^\dagger_{\dot{a}}(\mathcal{T}x), \quad T^{-1}\zeta_a(x)T = \zeta_T\zeta^a(\mathcal{T}x)$$

Majorana 旋量场的分解

下面讨论 **Majorana 旋量场**，**Majorana 条件**意味着 $\begin{pmatrix}\eta_a\\ \zeta^{\dagger\dot{a}}\end{pmatrix} = \psi = \mathcal{C}\bar{\psi}^{\mathrm{T}} = \begin{pmatrix}\zeta_a\\ \eta^{\dagger\dot{a}}\end{pmatrix}$

即 $\eta = \zeta$，这表明 Majorana 旋量场中的左手和右手 Weyl 旋量场是相关的

因此，可以将 Majorana 旋量场 $\psi(x)$ 分解成

$$\psi(x) = \begin{pmatrix}\eta_a(x)\\ \eta^{\dagger\dot{a}}(x)\end{pmatrix}$$

Majorana 旋量场的分解

下面讨论 **Majorana 旋量场**，**Majorana 条件**意味着 $\begin{pmatrix}\eta_a\\ \zeta^{\dagger\dot{a}}\end{pmatrix} = \psi = \mathcal{C}\bar{\psi}^{\mathrm{T}} = \begin{pmatrix}\zeta_a\\ \eta^{\dagger\dot{a}}\end{pmatrix}$

即 $\eta = \zeta$，这表明 Majorana 旋量场中的**左手**和**右手** Weyl 旋量场是**相关**的

因此，可以将 Majorana 旋量场 $\psi(x)$ 分解成 $\psi(x) = \begin{pmatrix}\eta_a(x)\\ \eta^{\dagger\dot{a}}(x)\end{pmatrix}$

而**自由 Majorana 旋量场**的**拉氏量**分解为

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = \frac{1}{2}\begin{pmatrix}\eta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}-m\delta_a{}^b & \mathrm{i}(\sigma^\mu)_{a\dot{b}}\partial_\mu\\ \mathrm{i}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu & -m\delta^{\dot{a}}{}_{\dot{b}}\end{pmatrix}\begin{pmatrix}\eta_b\\ \eta^{\dagger\dot{b}}\end{pmatrix}\\ &= \frac{1}{2}[\mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta + \mathrm{i}\eta\sigma^\mu\partial_\mu\eta^\dagger - m(\eta\eta + \eta^\dagger\eta^\dagger)]\end{aligned}$$

利用 $\zeta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\zeta$ 将方括号中**第二项**化为

$$\mathrm{i}\eta\sigma^\mu\partial_\mu\eta^\dagger = \mathrm{i}\partial_\mu(\eta\sigma^\mu\eta^\dagger) - \mathrm{i}(\partial_\mu\eta)\sigma^\mu\eta^\dagger = \mathrm{i}\partial_\mu(\eta\sigma^\mu\eta^\dagger) + \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta$$

扔掉全散度项 $\mathrm{i}\partial_\mu(\eta\sigma^\mu\eta^\dagger)$，拉氏量变成 $\mathcal{L} = \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta - \frac{1}{2}m(\eta\eta + \eta^\dagger\eta^\dagger)$

这里的质量项只涉及**一个 Weyl 旋量场** $\eta_a(x)$，称为 **Majorana 质量项**

Majorana 旋量场的 $\bar{\psi}\gamma^\mu\psi$ 和 $\bar{\psi}\sigma^{\mu\nu}\psi$

- $\zeta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\zeta$、$\eta\sigma^\mu\bar{\sigma}^\nu\zeta = \zeta\sigma^\nu\bar{\sigma}^\mu\eta$ 和 $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger = \zeta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger$ 意味着

$$\eta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\eta, \quad \eta\sigma^\mu\bar{\sigma}^\nu\eta = \eta\sigma^\nu\bar{\sigma}^\mu\eta, \quad \eta^\dagger\bar{\sigma}^\mu\sigma^\nu\eta^\dagger = \eta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger$$

- 对于 **Majorana 旋量场**，$\eta = \zeta$，$\bar{\psi}\gamma^\mu\psi = \zeta\sigma^\mu\zeta^\dagger + \eta^\dagger\bar{\sigma}^\mu\eta$ 化为

$$\bar{\psi}\gamma^\mu\psi = \eta\sigma^\mu\eta^\dagger + \eta^\dagger\bar{\sigma}^\mu\eta = -\eta^\dagger\bar{\sigma}^\mu\eta + \eta^\dagger\bar{\sigma}^\mu\eta = 0$$

- $\bar{\psi}\sigma^{\mu\nu}\psi = \frac{\mathrm{i}}{2}\zeta(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)\eta + \frac{\mathrm{i}}{2}\eta^\dagger(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)\zeta^\dagger$ 化为

$$\bar{\psi}\sigma^{\mu\nu}\psi = \frac{\mathrm{i}}{2}(\eta\sigma^\mu\bar{\sigma}^\nu\eta - \eta\sigma^\nu\bar{\sigma}^\mu\eta) + \frac{\mathrm{i}}{2}(\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\eta^\dagger - \eta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger) = 0$$

- 这样就验证了 9.2.2 小节的**结论**

9.7 节　Majorana 旋量场相关 Feynman 规则

- 7.1.1 小节提到，由于 **Dirac 旋量场**可以携带某种 **U(1) 荷**，相应**费米子线**上的**箭头**代表 $U(1)$ 荷流动的方向，或者说**费米子数**流动的方向

- 另一方面，**Majorana 旋量场不能**携带任何 $\mathrm{U}(1)$ 荷，不存在费米子数流动的方向，相应的费米子线则**不应该具备箭头**

- 如果相互作用过程涉及到 Majorana 旋量场与 Dirac 旋量场的耦合，**带箭头**与**不带箭头**的费米子线将在顶点处**交汇**，导致**费米子数破坏** (fermion-number violation)

- 我们需要研究适用于这种情况的 Feynman 规则

- 本节讨论一个简单例子，更一般的情况可参考**文献**

- A. Denner, H. Eck, O. Hahn, and J. Kublbeck, “Feynman rules for fermion number violating interactions,” Nucl. Phys. B 387 (1992) 467–481

9.7.1 小节　拉氏量和 CP 对称性

- 考虑**复标量场** $\phi(x)$、**Dirac 旋量场** $\psi(x)$ 和 **Majorana 旋量场** $\chi(x)$ 构成拉氏量

$$\mathcal{L} = (\partial^\mu \phi^\dagger)\partial_\mu \phi - m_\phi^2 \phi^\dagger \phi + \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m_\psi)\psi + \frac{1}{2}\bar{\chi}(\mathrm{i}\gamma^\mu \partial_\mu - m_\chi)\chi + \mathcal{L}_{\mathrm{int}}$$

- **相互作用拉氏量**为 $\mathcal{L}_{\mathrm{int}} = \kappa\, \phi^\dagger \bar{\chi} P_{\mathrm{R}} \psi + \kappa^* \phi \bar{\psi} P_{\mathrm{L}} \chi$

- κ 是一个**复耦合常数**，$\mathcal{L}_{\mathrm{int}}$ 是**厄米**的，因为 $\mathcal{L}_{\mathrm{int}}$ 中两项互为厄米共轭，

$$(\kappa\, \phi^\dagger \bar{\chi} P_{\mathrm{R}} \psi)^\dagger = \kappa^* \psi^\dagger P_{\mathrm{R}} \gamma^0 \chi = \kappa^* \psi^\dagger \gamma^0 P_{\mathrm{L}} \chi = \kappa^* \phi \bar{\psi} P_{\mathrm{L}} \chi$$

- 这样的相互作用涉及一个标量场和两个旋量场，属于 **Yukawa 相互作用**

9.7.1 小节 拉氏量和 CP 对称性

考虑**复标量场** $\phi(x)$、**Dirac 旋量场** $\psi(x)$ 和 **Majorana 旋量场** $\chi(x)$ 构成拉氏量

$$\mathcal{L} = (\partial^\mu \phi^\dagger)\partial_\mu \phi - m_\phi^2 \phi^\dagger \phi + \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m_\psi)\psi + \frac{1}{2}\bar{\chi}(\mathrm{i}\gamma^\mu \partial_\mu - m_\chi)\chi + \mathcal{L}_{\mathrm{int}}$$

相互作用拉氏量为 $\mathcal{L}_{\mathrm{int}} = \kappa\,\phi^\dagger \bar{\chi} P_{\mathrm{R}} \psi + \kappa^* \phi \bar{\psi} P_{\mathrm{L}} \chi$

κ 是一个**复耦合常数**，$\mathcal{L}_{\mathrm{int}}$ 是**厄米**的，因为 $\mathcal{L}_{\mathrm{int}}$ 中两项互为厄米共轭，

$$(\kappa\,\phi^\dagger \bar{\chi} P_{\mathrm{R}} \psi)^\dagger = \kappa^* \psi^\dagger P_{\mathrm{R}} \gamma^0 \chi = \kappa^* \psi^\dagger \gamma^0 P_{\mathrm{L}} \chi = \kappa^* \phi \bar{\psi} P_{\mathrm{L}} \chi$$

这样的相互作用涉及一个标量场和两个旋量场，属于 **Yukawa 相互作用**

作 U(1) 整体变换 $\phi'(x) = e^{iq\theta}\phi(x)$ 和 $\psi'(x) = e^{iq\theta}\psi(x)$，则拉氏量 $\mathcal{L}$ 不变

可见，这个理论具有一个 **U(1) 整体对称性**，而**复标量场** $\phi(x)$ 和 **Dirac 旋量场** $\psi(x)$ 的 **U(1) 荷相同**，均为 q

将**耦合常数**分解为**实部**和**虚部**，$\kappa = \kappa_{\mathrm{R}} + \mathrm{i}\kappa_{\mathrm{I}}$，则相互作用拉氏量化为

$$\mathcal{L}_{\mathrm{int}} = \kappa_{\mathrm{R}}(\phi^\dagger \bar{\chi} P_{\mathrm{R}} \psi + \phi \bar{\psi} P_{\mathrm{L}} \chi) + \kappa_{\mathrm{I}}(\mathrm{i}\phi^\dagger \bar{\chi} P_{\mathrm{R}} \psi - \mathrm{i}\phi \bar{\psi} P_{\mathrm{L}} \chi)$$

C 破坏和 P 破坏

假设三个量子场的 C、P 变换为

$$C^{-1}\phi(x)C = \eta_C^*\phi^\dagger(x), \quad C^{-1}\psi(x)C = \zeta_C^*\,\mathcal{C}\bar{\psi}^{\mathrm{T}}(x), \quad C^{-1}\chi(x)C = \tilde{\zeta}_C^*\chi(x)$$

$$P^{-1}\phi(x)P = \eta_P^*\phi(\mathcal{P}x), \quad P^{-1}\psi(x)P = \zeta_P^*\gamma^0\psi(\mathcal{P}x), \quad P^{-1}\chi(x)P = \tilde{\zeta}_P^*\gamma^0\chi(\mathcal{P}x)$$

推出**算符** $\phi^\dagger\bar{\chi}P_{\mathrm{R}}\psi$ 的 C、P 变换

$$C^{-1}\phi^\dagger(x)\bar{\chi}(x)P_{\mathrm{R}}\psi(x)C = \eta_C\zeta_C^*\tilde{\zeta}_C\phi(x)\bar{\psi}(x)P_{\mathrm{R}}\chi(x)$$

$$P^{-1}\phi^\dagger(x)\bar{\chi}(x)P_{\mathrm{R}}\psi(x)P = \eta_P\zeta_P^*\tilde{\zeta}_P\phi^\dagger(\mathcal{P}x)\bar{\chi}(\mathcal{P}x)P_{\mathrm{L}}\psi(\mathcal{P}x)$$

而**算符** $\phi\bar{\psi}P_{\mathrm{L}}\chi$ 的 C、P 变换为

$$C^{-1}\phi(x)\bar{\psi}(x)P_{\mathrm{L}}\chi(x)C = \eta_C^*\zeta_C\tilde{\zeta}_C^*\phi^\dagger(x)\bar{\chi}(x)P_{\mathrm{L}}\psi(x)$$

$$P^{-1}\phi(x)\bar{\psi}(x)P_{\mathrm{L}}\chi(x)P = \eta_P^*\zeta_P\tilde{\zeta}_P^*\phi(\mathcal{P}x)\bar{\psi}(\mathcal{P}x)P_{\mathrm{R}}\chi(\mathcal{P}x)$$

C 破坏和 P 破坏

假设三个量子场的 C、P 变换为

$$C^{-1}\phi(x)C = \eta_C^*\phi^\dagger(x),\quad C^{-1}\psi(x)C = \zeta_C^*\,\mathcal{C}\bar{\psi}^{\mathrm{T}}(x),\quad C^{-1}\chi(x)C = \tilde{\zeta}_C^*\chi(x)$$

$$P^{-1}\phi(x)P = \eta_P^*\phi(\mathcal{P}x),\quad P^{-1}\psi(x)P = \zeta_P^*\gamma^0\psi(\mathcal{P}x),\quad P^{-1}\chi(x)P = \tilde{\zeta}_P^*\gamma^0\chi(\mathcal{P}x)$$

推出**算符** $\phi^\dagger\bar{\chi}P_{\mathrm{R}}\psi$ 的 C、P 变换

$$C^{-1}\phi^\dagger(x)\bar{\chi}(x)P_{\mathrm{R}}\psi(x)C = \eta_C\zeta_C^*\tilde{\zeta}_C\phi(x)\bar{\psi}(x)P_{\mathrm{R}}\chi(x)$$

$$P^{-1}\phi^\dagger(x)\bar{\chi}(x)P_{\mathrm{R}}\psi(x)P = \eta_P\zeta_P^*\tilde{\zeta}_P\phi^\dagger(\mathcal{P}x)\bar{\chi}(\mathcal{P}x)P_{\mathrm{L}}\psi(\mathcal{P}x)$$

而**算符** $\phi\bar{\psi}P_{\mathrm{L}}\chi$ 的 C、P 变换为

$$C^{-1}\phi(x)\bar{\psi}(x)P_{\mathrm{L}}\chi(x)C = \eta_C^*\zeta_C\tilde{\zeta}_C^*\phi^\dagger(x)\bar{\chi}(x)P_{\mathrm{L}}\psi(x)$$

$$P^{-1}\phi(x)\bar{\psi}(x)P_{\mathrm{L}}\chi(x)P = \eta_P^*\zeta_P\tilde{\zeta}_P^*\phi(\mathcal{P}x)\bar{\psi}(\mathcal{P}x)P_{\mathrm{R}}\chi(\mathcal{P}x)$$

无论作 ***C* 变换**还是 ***P* 变换**，**相互作用拉氏量** $\mathcal{L}_{\mathrm{int}} = \kappa\,\phi^\dagger\bar{\chi}P_{\mathrm{R}}\psi + \kappa^*\phi\bar{\psi}P_{\mathrm{L}}\chi$ 都不能保持不变，因此理论**不具有**电荷共轭对称性和空间反射对称性

换言之，这个理论既是 ***C* 破坏** (C-violation) 的，又是 ***P* 破坏** (P-violation) 的

CP 破坏?

- 进一步，算符 $\phi^\dagger\bar{\chi}P_{\mathrm{R}}\psi$ 和 $\phi\bar{\psi}P_{\mathrm{L}}\chi$ 的 CP 变换为

$$(CP)^{-1}\phi^\dagger(x)\bar{\chi}(x)P_{\mathrm{R}}\psi(x)CP = \eta_{CP}\phi(\mathcal{P}x)\bar{\psi}(\mathcal{P}x)P_{\mathrm{L}}\chi(\mathcal{P}x)$$

$$(CP)^{-1}\phi(x)\bar{\psi}(x)P_{\mathrm{L}}\chi(x)CP = \eta_{CP}^*\phi^\dagger(\mathcal{P}x)\bar{\chi}(\mathcal{P}x)P_{\mathrm{R}}\psi(\mathcal{P}x)$$

- 其中 $\eta_{CP} \equiv \eta_C\eta_P^*\zeta_C^*\zeta_P\tilde{\zeta}_C\tilde{\zeta}_P^*$
- 9.1.1 小节末提到，**复场**的分立变换**相位因子**的取值是**任意**的
- 如果**适当选取** $\phi(x)$ 和 $\psi(x)$ **相位因子**的值，使得 $\eta_{CP} = \eta_{CP}^* = +1$
- 则算符 $\phi^\dagger\bar{\chi}P_{\mathrm{R}}\psi + \phi\bar{\psi}P_{\mathrm{L}}\chi$ 在 CP 变换下**不变**
- 而相互作用拉氏量 $\mathcal{L}_{\mathrm{int}} = \kappa_{\mathrm{R}}(\phi^\dagger\bar{\chi}P_{\mathrm{R}}\psi + \phi\bar{\psi}P_{\mathrm{L}}\chi) + \kappa_{\mathrm{I}}(\mathrm{i}\phi^\dagger\bar{\chi}P_{\mathrm{R}}\psi - \mathrm{i}\phi\bar{\psi}P_{\mathrm{L}}\chi)$ 中

κ_{R} **对应的项**具有 ***CP* 对称性**，κ_{I} **对应的项**引起 ***CP* 破坏** (CP-violation)

CP 破坏?

进一步，算符 $\phi^\dagger\bar{\chi}P_\mathrm{R}\psi$ 和 $\phi\bar{\psi}P_\mathrm{L}\chi$ 的 CP 变换为

$$(CP)^{-1}\phi^\dagger(x)\bar{\chi}(x)P_\mathrm{R}\psi(x)CP = \eta_{CP}\phi(\mathcal{P}x)\bar{\psi}(\mathcal{P}x)P_\mathrm{L}\chi(\mathcal{P}x)$$

$$(CP)^{-1}\phi(x)\bar{\psi}(x)P_\mathrm{L}\chi(x)CP = \eta_{CP}^*\phi^\dagger(\mathcal{P}x)\bar{\chi}(\mathcal{P}x)P_\mathrm{R}\psi(\mathcal{P}x)$$

其中 $\eta_{CP} \equiv \eta_C\eta_P^*\zeta_C^*\zeta_P\tilde{\zeta}_C\tilde{\zeta}_P^*$

9.1.1 小节末提到，**复场**的分立变换**相位因子**的取值是**任意**的

如果**适当选取** $\phi(x)$ 和 $\psi(x)$ **相位因子**的值，使得 $\eta_{CP} = \eta_{CP}^* = +1$

则算符 $\phi^\dagger\bar{\chi}P_\mathrm{R}\psi + \phi\bar{\psi}P_\mathrm{L}\chi$ 在 CP 变换下**不变**

而相互作用拉氏量 $\mathcal{L}_\mathrm{int} = \kappa_\mathrm{R}(\phi^\dagger\bar{\chi}P_\mathrm{R}\psi + \phi\bar{\psi}P_\mathrm{L}\chi) + \kappa_\mathrm{I}(\mathrm{i}\phi^\dagger\bar{\chi}P_\mathrm{R}\psi - \mathrm{i}\phi\bar{\psi}P_\mathrm{L}\chi)$ 中 κ_R **对应的项**具有 ***CP* 对称性**，κ_I **对应的项**引起 ***CP* 破坏** (CP-violation)

如果**相位因子的取值**使得 $\eta_{CP} = \eta_{CP}^* = -1$

则算符 $\mathrm{i}\phi^\dagger\bar{\chi}P_\mathrm{R}\psi - \mathrm{i}\phi\bar{\psi}P_\mathrm{L}\chi$ 在 CP 变换下**不变**

而 κ_I **对应的项**具有 ***CP* 对称性**，κ_R **对应的项**引起 ***CP* 破坏**

⁉️ 因此，当 $\kappa_\mathrm{R} \neq 0$ **且** $\kappa_\mathrm{I} \neq 0$ 时，相互作用拉氏量 $\mathcal{L}_\mathrm{int}$ **看起来会破坏 *CP* 对称性**

CP 对称性

🦒 不过，**Dirac 旋量场** $\psi(x)$ 是**复**的量子场，即 Hilbert 空间中的**非自共轭**算符，它的**相位**具有**任意性**，可用于**吸收耦合常数** $\kappa \equiv |\kappa|e^{-i\varphi}$ 的**相位** φ

⚙️ 如果将 Dirac 旋量场**重新定义**为 $\psi'(x) = e^{-i\varphi}\psi(x)$，则 $\bar{\psi}'(x) = e^{i\varphi}\psi(x)$，于是

$$\mathcal{L}_{\text{int}} = |\kappa|e^{-i\varphi}\,\phi^\dagger\bar{\chi}P_{\text{R}}\psi + |\kappa|e^{i\varphi}\phi\bar{\psi}P_{\text{L}}\chi = |\kappa|(\phi^\dagger\bar{\chi}P_{\text{R}}\psi' + \phi\bar{\psi}'P_{\text{L}}\chi)$$ 描述**同一个理论**

⚖️ 但此时**耦合常数** $|\kappa|$ 是**实数**，不会引起 CP 破坏

❗ 因此，这个理论实际上是具有 ***CP* 对称性**的

‼️ 当理论中**所有复耦合常数的相位不能完全被复场吸收**时，才会出现 ***CP* 破坏**

CP 对称性

🦒 不过，**Dirac 旋量场** $\psi(x)$ 是**复**的量子场，即 Hilbert 空间中的**非自共轭**算符，它的**相位**具有**任意性**，可用于**吸收耦合常数** $\kappa \equiv |\kappa|e^{-i\varphi}$ 的**相位** φ

⚙️ 如果将 Dirac 旋量场**重新定义**为 $\psi'(x) = e^{-i\varphi}\psi(x)$，则 $\bar{\psi}'(x) = e^{i\varphi}\psi(x)$，于是

$\mathcal{L}_{\text{int}} = |\kappa|e^{-i\varphi}\,\phi^\dagger\bar{\chi}P_R\psi + |\kappa|e^{i\varphi}\phi\bar{\psi}P_L\chi = |\kappa|(\phi^\dagger\bar{\chi}P_R\psi' + \phi\bar{\psi}'P_L\chi)$ 描述**同一个理论**

⚖️ 但此时**耦合常数** $|\kappa|$ 是**实数**，不会引起 CP 破坏

❗ 因此，这个理论实际上是具有 **CP 对称性**的

‼️ 当理论中**所有复耦合常数的相位不能完全被复场吸收**时，才会出现 **CP 破坏**

🔩 另一方面，像实标量场、实矢量场和 Majorana 旋量场这样的**实场**必须满足**自共轭条件**，这导致它**不具有**相位任意性

🗜️ 在下面的讨论中，不失一般性，将**耦合常数** κ 取为**实数**，相互作用拉氏量表达为

$$\mathcal{L}_{\text{int}} = \kappa(\phi^\dagger\bar{\chi}\Gamma_1\psi + \phi\bar{\psi}\Gamma_2\chi)$$

🔗 这里引入了 $\Gamma_1 = P_R$ 和 $\Gamma_2 = P_L$，下面许多结论与 Γ_1 和 Γ_2 的具体形式**无关**

9.7.2 小节 Feynman 规则

将 **Dirac 旋量场**、**复标量场**和 **Majorana 旋量场**的平面波展开式表达为

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_\lambda [u(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x} + v(\mathbf{p},\lambda) b^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x}]$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (c_{\mathbf{p}} e^{-ip\cdot x} + d^\dagger_{\mathbf{p}} e^{ip\cdot x})$$

$$\chi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_\lambda [u(\mathbf{p},\lambda) f_{\mathbf{p},\lambda} e^{-ip\cdot x} + v(\mathbf{p},\lambda) f^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x}]$$

相应地，引入以下单粒子态，

Dirac 正费米子 ψ 的单粒子态 $|\mathbf{p}^+,\lambda\rangle = \sqrt{2E_{\mathbf{p}}}\, a^\dagger_{\mathbf{p},\lambda}\, |0\rangle$

Dirac 反费米子 $\bar{\psi}$ 的单粒子态 $|\mathbf{p}^-,\lambda\rangle = \sqrt{2E_{\mathbf{p}}}\, b^\dagger_{\mathbf{p},\lambda}\, |0\rangle$

正标量玻色子 ϕ 的单粒子态 $|\mathbf{p}^+\rangle = \sqrt{2E_{\mathbf{p}}}\, c^\dagger_{\mathbf{p}}\, |0\rangle$

反标量玻色子 $\bar{\phi}$ 的单粒子态 $|\mathbf{p}^-\rangle = \sqrt{2E_{\mathbf{p}}}\, d^\dagger_{\mathbf{p}}\, |0\rangle$

Majorana 费米子 χ 的单粒子态 $|\mathbf{p},\lambda\rangle = \sqrt{2E_{\mathbf{p}}}\, f^\dagger_{\mathbf{p},\lambda}\, |0\rangle$

注意，**Majorana 费米子** χ 是**纯中性**的，动量记号的右上角**没有正负号**

S 算符 $n = 1$ 阶

- Dirac 旋量场和复标量场与初末态的缩并结果见第 7 章

- Majorana 旋量场与初末态的缩并定义为

$$
\begin{aligned}
\langle 0|\, \overbracket{\chi(x)\, |\mathbf{p}}, \lambda\rangle &\equiv \langle 0|\, \chi^{(+)}(x)\, |\mathbf{p}, \lambda\rangle = u(\mathbf{p}, \lambda) e^{-ip\cdot x} \\
\langle 0|\, \overbracket{\bar{\chi}(x)\, |\mathbf{p}}, \lambda\rangle &\equiv \langle 0|\, \bar{\chi}^{(+)}(x)\, |\mathbf{p}, \lambda\rangle = \bar{v}(\mathbf{p}, \lambda) e^{-ip\cdot x} \\
\langle \overbracket{\mathbf{p}, \lambda|\, \bar{\chi}}(x)\, |0\rangle &\equiv \langle \mathbf{p}, \lambda|\, \bar{\chi}^{(-)}(x)\, |0\rangle = \bar{u}(\mathbf{p}, \lambda) e^{ip\cdot x} \\
\langle \overbracket{\mathbf{p}, \lambda|\, \chi}(x)\, |0\rangle &\equiv \langle \mathbf{p}, \lambda|\, \chi^{(-)}(x)\, |0\rangle = v(\mathbf{p}, \lambda) e^{ip\cdot x}
\end{aligned}
$$

S 算符 $n=1$ 阶

- **Dirac 旋量场**和**复标量场**与**初末态**的**缩并**结果见**第 7 章**

- **Majorana 旋量场**与**初末态**的**缩并**定义为

$$
\begin{aligned}
\langle 0|\,\overbracket{\chi(x)\,|\mathbf{p}},\lambda\rangle &\equiv \langle 0|\,\chi^{(+)}(x)\,|\mathbf{p},\lambda\rangle = u(\mathbf{p},\lambda)\mathrm{e}^{-\mathrm{i}p\cdot x}\\
\langle 0|\,\overbracket{\bar{\chi}(x)\,|\mathbf{p}},\lambda\rangle &\equiv \langle 0|\,\bar{\chi}^{(+)}(x)\,|\mathbf{p},\lambda\rangle = \bar{v}(\mathbf{p},\lambda)\mathrm{e}^{-\mathrm{i}p\cdot x}\\
\langle \overbracket{\mathbf{p},\lambda|\,\bar{\chi}}(x)\,|0\rangle &\equiv \langle \mathbf{p},\lambda|\,\bar{\chi}^{(-)}(x)\,|0\rangle = \bar{u}(\mathbf{p},\lambda)\mathrm{e}^{\mathrm{i}p\cdot x}\\
\langle \overbracket{\mathbf{p},\lambda|\,\chi}(x)\,|0\rangle &\equiv \langle \mathbf{p},\lambda|\,\chi^{(-)}(x)\,|0\rangle = v(\mathbf{p},\lambda)\mathrm{e}^{\mathrm{i}p\cdot x}
\end{aligned}
$$

- 由于**相互作用哈密顿量密度** $\mathcal{H}_1=-\mathcal{L}_{\text{int}}$，$\mathrm{i}T$ **算符**展开式中 $n=1$ 的项为

$$
\begin{aligned}
\mathrm{i}T^{(1)} &= -\mathrm{i}\int d^4x\,\mathsf{T}[\mathcal{H}_1(x)] = \mathrm{i}\int d^4x\,\mathsf{T}[\mathcal{L}_{\text{int}}(x)]\\
&= \mathrm{i}\kappa\int d^4x\,\mathsf{T}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)+\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)]
\end{aligned}
$$

- 根据 **Wick 定理**，$\mathrm{i}T^{(1)}$ 只包含下面两项，

$$
\mathrm{i}T_1^{(1)} = \mathrm{i}\kappa\int d^4x\,\mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)],\quad \mathrm{i}T_2^{(1)} = \mathrm{i}\kappa\int d^4x\,\mathsf{N}[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)]
$$

$\psi \to \chi\phi$ 衰变过程

考虑 $\psi \to \chi\phi$ **衰变**，初末态为 $|\mathbf{p}^+, \lambda\rangle$ 和 $|\mathbf{q}, \lambda'; \mathbf{k}^+\rangle$，$\mathrm{i}T_1^{(1)}$ 贡献的 T 矩阵元是

$$\begin{aligned}\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathrm{i}T_1^{(1)} | \mathbf{p}^+, \lambda\rangle &= \mathrm{i}\kappa \int d^4x \, \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)] | \mathbf{p}^+, \lambda\rangle \\ &= \mathrm{i}\kappa \int d^4x \, \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)] | \mathbf{p}^+, \lambda\rangle \\ &= \mathrm{i}\kappa \int d^4x \, \bar{u}(\mathbf{q}, \lambda')\Gamma_1 u(\mathbf{p}, \lambda) e^{-\mathrm{i}(p-q-k)\cdot x} \\ &= \mathrm{i}\kappa \, \bar{u}(\mathbf{q}, \lambda')\Gamma_1 u(\mathbf{p}, \lambda) \, (2\pi)^4 \delta^{(4)}(p-q-k)\end{aligned}$$

这是计算 T 矩阵元的**第一种方法**，与 **7.1 节**介绍的方法一样

$\psi \to \chi\phi$ 衰变过程

- 考虑 $\psi \to \chi\phi$ **衰变**，初末态为 $|\mathbf{p}^+, \lambda\rangle$ 和 $|\mathbf{q}, \lambda'; \mathbf{k}^+\rangle$，$\mathrm{i}T_1^{(1)}$ 贡献的 T 矩阵元是

$$\begin{aligned}
\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathrm{i}T_1^{(1)} |\mathbf{p}^+, \lambda\rangle &= \mathrm{i}\kappa \int \mathrm{d}^4x \, \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)] |\mathbf{p}^+, \lambda\rangle \\
&= \mathrm{i}\kappa \int \mathrm{d}^4x \, \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)] |\mathbf{p}^+, \lambda\rangle \\
&= \mathrm{i}\kappa \int \mathrm{d}^4x \, \bar{u}(\mathbf{q}, \lambda')\Gamma_1 u(\mathbf{p}, \lambda) e^{-\mathrm{i}(p-q-k)\cdot x} \\
&= \mathrm{i}\kappa \, \bar{u}(\mathbf{q}, \lambda')\Gamma_1 u(\mathbf{p}, \lambda) \, (2\pi)^4 \delta^{(4)}(p-q-k)
\end{aligned}$$

- 这是计算 T 矩阵元的**第一种方法**，与 **7.1 节**介绍的方法一样
- 利用**电荷共轭变换**，可以引进**第二种计算方法**
- 将相互作用算符 $\bar{\chi}\Gamma_1\psi$ 化为

$$\begin{aligned}
\bar{\chi}\Gamma_1\psi &= (\bar{\chi}\Gamma_1\psi)^{\mathrm{T}} = -\psi^{\mathrm{T}}\Gamma_1^{\mathrm{T}}\bar{\chi}^{\mathrm{T}} = -\psi^{\mathrm{T}}\mathcal{C}^{-1}\mathcal{C}\Gamma_1^{\mathrm{T}}\mathcal{C}^{-1}\mathcal{C}\bar{\chi}^{\mathrm{T}} \\
&= \psi^{\mathrm{T}}\mathcal{C}\mathcal{C}\Gamma_1^{\mathrm{T}}\mathcal{C}^{-1}\mathcal{C}\bar{\chi}^{\mathrm{T}} = \bar{\psi}^{\mathrm{C}}\Gamma_1^{\mathrm{C}}\chi^{\mathrm{C}}
\end{aligned}$$

- 同理推出 $\bar{\psi}\Gamma_2\chi = \bar{\chi}^{\mathrm{C}}\Gamma_2^{\mathrm{C}}\psi^{\mathrm{C}}$

第二种计算方法

- 通过 **Majorana 条件** $\chi=\chi^{\mathrm{C}}$ 将 $\bar{\chi}\Gamma_1\psi=\bar{\psi}^{\mathrm{C}}\Gamma_1^{\mathrm{C}}\chi^{\mathrm{C}}$ 和 $\bar{\psi}\Gamma_2\chi=\bar{\chi}^{\mathrm{C}}\Gamma_2^{\mathrm{C}}\psi^{\mathrm{C}}$ 化为

$$\bar{\chi}\Gamma_1\psi=\bar{\psi}^{\mathrm{C}}\Gamma_1^{\mathrm{C}}\chi,\quad \bar{\psi}\Gamma_2\chi=\bar{\chi}\Gamma_2^{\mathrm{C}}\psi^{\mathrm{C}}$$

- 从而将 $\mathrm{i}T_1^{(1)}$ 和 $\mathrm{i}T_2^{(1)}$ 改写为

$$\begin{aligned}
\mathrm{i}T_1^{(1)} &= \mathrm{i}\kappa\int \mathrm{d}^4x\,\mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)]=\mathrm{i}\kappa\int \mathrm{d}^4x\,\mathsf{N}[\phi^\dagger(x)\bar{\psi}^{\mathrm{C}}(x)\Gamma_1^{\mathrm{C}}\chi(x)]\\
\mathrm{i}T_2^{(1)} &= \mathrm{i}\kappa\int \mathrm{d}^4x\,\mathsf{N}[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)]=\mathrm{i}\kappa\int \mathrm{d}^4x\,\mathsf{N}[\phi(x)\bar{\chi}(x)\Gamma_2^{\mathrm{C}}\psi^{\mathrm{C}}(x)]
\end{aligned}$$

- 注意，此时**旋量场算符排列的次序**与原来**相反**
- 现在，$\mathrm{i}T_1^{(1)}$ 贡献的 $\psi\to\chi\phi$ **过程** T **矩阵元**也可以表达成

$$\begin{aligned}
&\langle\mathbf{q},\lambda';\mathbf{k}^+|\,\mathrm{i}T_1^{(1)}\,|\mathbf{p}^+,\lambda\rangle\\
&=\mathrm{i}\kappa\int \mathrm{d}^4x\,\langle\mathbf{q},\lambda';\mathbf{k}^+|\,\mathsf{N}[\phi^\dagger(x)\bar{\psi}_a^{\mathrm{C}}(x)(\Gamma_1^{\mathrm{C}})_{ab}\chi_b(x)]\,|\mathbf{p}^+,\lambda\rangle\\
&=-\mathrm{i}\kappa\int \mathrm{d}^4x\,\langle\mathbf{q},\lambda';\mathbf{k}^+|\,\phi^{\dagger(-)}(x)\chi_b^{(-)}(x)(\Gamma_1^{\mathrm{C}})_{ab}\bar{\psi}_a^{\mathrm{C}(+)}(x)\,|\mathbf{p}^+,\lambda\rangle
\end{aligned}$$

电荷共轭场 $\psi^{\mathrm{C}}(x)$ 的平面波展开和初末态缩并

Dirac 旋量场 $\psi(x)$ 的**电荷共轭场** $\psi^{\mathrm{C}}(x)$ 的平面波展开式是

$$\psi^{\mathrm{C}}(x) = \mathcal{C}\bar{\psi}^{\mathrm{T}} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda\left[\mathcal{C}\bar{v}^{\mathrm{T}}(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x} + \mathcal{C}\bar{u}^{\mathrm{T}}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda[u(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x} + v(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}]$$

跟 $\psi(x)$ **展开式**的差异只在于 a 与 b **互换**，相应 Dirac 共轭的展开式为

$$\bar{\psi}^{\mathrm{C}}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda[\bar{u}(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + \bar{v}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}]$$

据此，将电荷共轭场 $\psi^{\mathrm{C}}(x)$ 和 $\bar{\psi}^{\mathrm{C}}(x)$ 与**初末态**的**缩并**定义成

$$\langle 0|\,\psi^{\mathrm{C}}(x)\,|\mathbf{p}^-,\lambda\rangle \equiv \langle 0|\,\psi^{\mathrm{C}(+)}(x)\,|\mathbf{p}^-,\lambda\rangle = u(\mathbf{p},\lambda)e^{-ip\cdot x}$$

$$\langle 0|\,\bar{\psi}^{\mathrm{C}}(x)\,|\mathbf{p}^+,\lambda\rangle \equiv \langle 0|\,\bar{\psi}^{\mathrm{C}(+)}(x)\,|\mathbf{p}^+,\lambda\rangle = \bar{v}(\mathbf{p},\lambda)e^{-ip\cdot x}$$

$$\langle \mathbf{p}^-,\lambda|\,\bar{\psi}^{\mathrm{C}}(x)\,|0\rangle \equiv \langle \mathbf{p}^-,\lambda|\,\bar{\psi}^{\mathrm{C}(-)}(x)\,|0\rangle = \bar{u}(\mathbf{p},\lambda)e^{ip\cdot x}$$

$$\langle \mathbf{p}^+,\lambda|\,\psi^{\mathrm{C}}(x)\,|0\rangle \equiv \langle \mathbf{p}^+,\lambda|\,\psi^{\mathrm{C}(-)}(x)\,|0\rangle = v(\mathbf{p},\lambda)e^{ip\cdot x}$$

第二种方法的计算结果

$\psi \to \chi\phi$ 的 T 矩阵元变成

$$\begin{aligned}\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \,\mathrm{i}T_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= -\mathrm{i}\kappa \int \mathrm{d}^4x \, \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \, \mathsf{N}[\phi^\dagger(x)\chi_b(x)(\Gamma_1^{\mathrm{C}})_{ab}\bar{\psi}_a^{\mathrm{C}}(x)] \, | \mathbf{p}^+, \lambda \rangle \\ &= -\mathrm{i}\kappa \int \mathrm{d}^4x \, v_b(\mathbf{q}, \lambda')(\Gamma_1^{\mathrm{C}})_{ab}\bar{v}_a(\mathbf{p}, \lambda)\mathrm{e}^{-\mathrm{i}(p-q-k)\cdot x} \\ &= -\mathrm{i}\kappa \int \mathrm{d}^4x \, \bar{v}(\mathbf{p}, \lambda)\Gamma_1^{\mathrm{C}} v(\mathbf{q}, \lambda')\mathrm{e}^{-\mathrm{i}(p-q-k)\cdot x} \\ &= -\mathrm{i}\kappa \, \bar{v}(\mathbf{p}, \lambda)\Gamma_1^{\mathrm{C}} v(\mathbf{q}, \lambda') \, (2\pi)^4\delta^{(4)}(p-q-k) \\ &= \mathrm{i}\kappa \int \mathrm{d}^4x \, \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \, \mathsf{N}[\phi^\dagger(x)\bar{\psi}^{\mathrm{C}}(x)\Gamma_1^{\mathrm{C}}\chi(x)] \, | \mathbf{p}^+, \lambda \rangle\end{aligned}$$

第二种方法的计算结果

$\psi \to \chi\phi$ 的 T 矩阵元变成

$$
\begin{aligned}
\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= -i\kappa \int d^4x \, \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\chi_b(x)(\Gamma_1^C)_{ab}\bar{\psi}_a^C(x)] | \mathbf{p}^+, \lambda \rangle \\
&= -i\kappa \int d^4x \, v_b(\mathbf{q}, \lambda')(\Gamma_1^C)_{ab}\bar{v}_a(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
&= -i\kappa \int d^4x \, \bar{v}(\mathbf{p}, \lambda)\Gamma_1^C v(\mathbf{q}, \lambda') e^{-i(p-q-k)\cdot x} \\
&= -i\kappa \, \bar{v}(\mathbf{p}, \lambda)\Gamma_1^C v(\mathbf{q}, \lambda') \, (2\pi)^4 \delta^{(4)}(p-q-k) \\
&= i\kappa \int d^4x \, \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\psi}^C(x)\Gamma_1^C \chi(x)] | \mathbf{p}^+, \lambda \rangle
\end{aligned}
$$

倒数第二行是**第二种方法**的计算结果，有

$$
\begin{aligned}
-\bar{v}(\mathbf{p}, \lambda)\Gamma_1^C v(\mathbf{q}, \lambda') &= -u^T(\mathbf{p}, \lambda)\,\mathcal{C}\Gamma_1^C\mathcal{C}\bar{u}^T(\mathbf{q}, \lambda') = u^T(\mathbf{p}, \lambda)\,\mathcal{C}\mathcal{C}^{-1}\Gamma_1^T\mathcal{C}\mathcal{C}^{-1}\bar{u}^T(\mathbf{q}, \lambda') \\
&= [u^T(\mathbf{p}, \lambda)\,\Gamma_1^T\bar{u}^T(\mathbf{q}, \lambda')]^T = \bar{u}(\mathbf{q}, \lambda')\Gamma_1 u(\mathbf{p}, \lambda)
\end{aligned}
$$

第二种方法结果与**第一种方法结果** $i\kappa\,\bar{u}(\mathbf{q}, \lambda')\Gamma_1 u(\mathbf{p}, \lambda)\,(2\pi)^4\delta^{(4)}(p-q-k)$ 相等

$\bar{\psi} \to \chi\bar{\phi}$ 衰变过程：第一种方法

另一方面，考虑 $\bar{\psi} \to \chi\bar{\phi}$ **衰变过程**，初态为 $|\mathbf{p}^-, \lambda\rangle$，末态为 $|\mathbf{q}, \lambda'; \mathbf{k}^-\rangle$

根据 $\mathrm{i}T_2^{(1)} = \mathrm{i}\kappa\int \mathrm{d}^4x\, \mathsf{N}[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)]$ 按**第一种方法**计算

$\mathrm{i}T_2^{(1)}$ 贡献的 T 矩阵元是

$$
\begin{aligned}
\langle\mathbf{q}, \lambda'; \mathbf{k}^-|\,\mathrm{i}T_2^{(1)}\,|\mathbf{p}^-, \lambda\rangle &= \mathrm{i}\kappa\int \mathrm{d}^4x\,\langle\mathbf{q}, \lambda'; \mathbf{k}^-|\,\mathsf{N}[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)]\,|\mathbf{p}^-, \lambda\rangle \\
&= \mathrm{i}\kappa\int \mathrm{d}^4x\,\langle\mathbf{q}, \lambda'; \mathbf{k}^-|\,\mathsf{N}[\phi(x)\bar{\psi}_a(x)(\Gamma_2)_{ab}\chi_b(x)]\,|\mathbf{p}^-, \lambda\rangle \\
&= -\mathrm{i}\kappa\int \mathrm{d}^4x\,\langle\mathbf{q}, \lambda'; \mathbf{k}^-|\,\mathsf{N}[\phi(x)\chi_b(x)(\Gamma_2)_{ab}\bar{\psi}_a(x)]\,|\mathbf{p}^-, \lambda\rangle \\
&= -\mathrm{i}\kappa\int \mathrm{d}^4x\, v_b(\mathbf{q}, \lambda')(\Gamma_2)_{ab}\bar{v}_a(\mathbf{p}, \lambda)\mathrm{e}^{-\mathrm{i}(p-q-k)\cdot x} \\
&= -\mathrm{i}\kappa\,\bar{v}(\mathbf{p}, \lambda)\Gamma_2 v(\mathbf{q}, \lambda')\,(2\pi)^4\delta^{(4)}(p-q-k)
\end{aligned}
$$

$\bar{\psi} \to \chi\bar{\phi}$ 衰变过程：第二种方法

- 根据 $\mathrm{i}T_2^{(1)} = \mathrm{i}\kappa \int \mathrm{d}^4x\, \mathsf{N}[\phi(x)\bar{\chi}(x)\Gamma_2^{\mathrm{C}}\psi^{\mathrm{C}}(x)]$ 按**第二种方法**计算

- $\mathrm{i}T_2^{(1)}$ 贡献的 T 矩阵元为

$$\begin{aligned}\langle \mathbf{q},\lambda';\mathbf{k}^-|\,\mathrm{i}T_2^{(1)}\,|\mathbf{p}^-,\lambda\rangle &= \mathrm{i}\kappa\int \mathrm{d}^4x\,\langle \mathbf{q},\lambda';\mathbf{k}^-|\,\mathsf{N}[\phi(x)\bar{\chi}(x)\Gamma_2^{\mathrm{C}}\psi^{\mathrm{C}}(x)]\,|\mathbf{p}^-,\lambda\rangle \\ &= \mathrm{i}\kappa\int \mathrm{d}^4x\,\langle \mathbf{q},\lambda';\mathbf{k}^-|\,\mathsf{N}[\phi(x)\bar{\chi}(x)\Gamma_2^{\mathrm{C}}\psi^{\mathrm{C}}(x)]\,|\mathbf{p}^-,\lambda\rangle \\ &= \mathrm{i}\kappa\int \mathrm{d}^4x\,\bar{u}(\mathbf{q},\lambda')\Gamma_2^{\mathrm{C}}u(\mathbf{p},\lambda)\mathrm{e}^{-\mathrm{i}(p-q-k)\cdot x} \\ &= \mathrm{i}\kappa\,\bar{u}(\mathbf{q},\lambda')\Gamma_2^{\mathrm{C}}u(\mathbf{p},\lambda)\,(2\pi)^4\delta^{(4)}(p-q-k)\end{aligned}$$

- 由于

$$\begin{aligned}\bar{u}(\mathbf{q},\lambda')\Gamma_2^{\mathrm{C}}u(\mathbf{p},\lambda) &= v^{\mathrm{T}}(\mathbf{q},\lambda')\,\mathcal{C}\Gamma_2^{\mathrm{C}}\mathcal{C}\bar{v}^{\mathrm{T}}(\mathbf{p},\lambda) = -v^{\mathrm{T}}(\mathbf{q},\lambda')\,\mathcal{C}\mathcal{C}^{-1}\Gamma_2^{\mathrm{T}}\mathcal{C}\mathcal{C}^{-1}\bar{v}^{\mathrm{T}}(\mathbf{p},\lambda) \\ &= -[v^{\mathrm{T}}(\mathbf{q},\lambda')\Gamma_2^{\mathrm{T}}\bar{v}^{\mathrm{T}}(\mathbf{p},\lambda)]^{\mathrm{T}} = -\bar{v}(\mathbf{p},\lambda)\Gamma_2 v(\mathbf{q},\lambda')\end{aligned}$$

- 两种方法的计算结果**相等**

费米子流方向

- 以上计算表明，**这两种方法**都是**有效**的，在实际计算中可采用任意一种方法

- 现在需要归纳出一套与这两种方法同时相容的 **Feynman 规则**，这样的规则将特别适用于处理费米子数破坏过程

- 为此，在每条连续费米子线附近添加一条带箭头的点划线，表示费米子流 (fermion flow) 的方向

- 费米子流的两种方向分别对应于上述**两种计算方法**

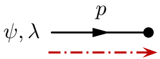
费米子流方向

- 以上计算表明，**这两种方法**都是**有效**的，在实际计算中可采用任意一种方法
- 现在需要归纳出一套与这两种方法**同时相容**的 **Feynman 规则**，这样的规则将特别适用于处理**费米子数破坏**过程
- 为此，在每条连续费米子线附近添加一条**带箭头的点划线**，表示**费米子流** (fermion flow) 的**方向**
- 费米子流的**两种方向**分别对应于上述**两种计算方法**
- 当**费米子流方向**与 Dirac 费米子**线上箭头方向相同**时，采用**第一种计算方法**
- 当**费米子流方向**与 Dirac 费米子**线上箭头方向相反**时，采用与**电荷共轭场**有关的**第二种计算方法**
- 这样一来，两种费米子流方向是**等价**的，对每条连续费米子线可采取**任意**一种方向进行计算

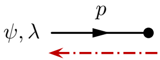
位置空间外线规则

于是，位置空间中费米子的外线规则如下，带箭头的点划线表示费米子流方向

1. Dirac 正费米子 ψ 入射外线：

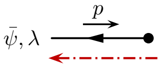


$$= \langle 0|\, \psi(x)\, |\mathbf{p}^+, \lambda\rangle = u(\mathbf{p}, \lambda)\mathrm{e}^{-\mathrm{i}p\cdot x}$$

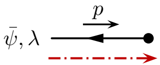


$$= \langle 0|\, \bar{\psi}^{\mathrm{C}}(x)\, |\mathbf{p}^+, \lambda\rangle = \bar{v}(\mathbf{p}, \lambda)\mathrm{e}^{-\mathrm{i}p\cdot x}$$

2. Dirac 反费米子 $\bar{\psi}$ 入射外线：

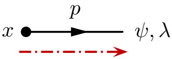


$$= \langle 0|\, \bar{\psi}(x)\, |\mathbf{p}^-, \lambda\rangle = \bar{v}(\mathbf{p}, \lambda)\mathrm{e}^{-\mathrm{i}p\cdot x}$$

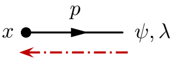


$$= \langle 0|\, \psi^{\mathrm{C}}(x)\, |\mathbf{p}^-, \lambda\rangle = u(\mathbf{p}, \lambda)\mathrm{e}^{-\mathrm{i}p\cdot x}$$

3. Dirac 正费米子 ψ 出射外线：



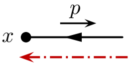
$$= \langle \mathbf{p}^+, \lambda|\, \bar{\psi}(x)\, |0\rangle = \bar{u}(\mathbf{p}, \lambda)\mathrm{e}^{\mathrm{i}p\cdot x}$$



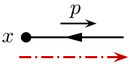
$$= \langle \mathbf{p}^+, \lambda|\, \psi^{\mathrm{C}}(x)\, |0\rangle = v(\mathbf{p}, \lambda)\mathrm{e}^{\mathrm{i}p\cdot x}$$

位置空间外线规则

4. **Dirac 反费米子 $\bar{\psi}$ 出射外线：**

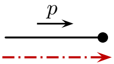


$$\bar{\psi},\lambda = \langle \mathbf{p}^-,\lambda|\,\psi(x)\,|0\rangle = v(\mathbf{p},\lambda)\mathrm{e}^{\mathrm{i}p\cdot x}$$

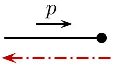


$$\bar{\psi},\lambda = \langle \mathbf{p}^-,\lambda|\,\bar{\psi}^{\mathrm{C}}(x)\,|0\rangle = \bar{u}(\mathbf{p},\lambda)\mathrm{e}^{\mathrm{i}p\cdot x}$$

5. **Majorana 费米子 χ 入射外线：**

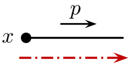


$$\chi,\lambda \quad x = \langle 0|\,\chi(x)\,|\mathbf{p},\lambda\rangle = u(\mathbf{p},\lambda)\mathrm{e}^{-\mathrm{i}p\cdot x}$$

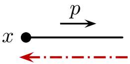


$$\chi,\lambda \quad x = \langle 0|\,\bar{\chi}(x)\,|\mathbf{p},\lambda\rangle = \bar{v}(\mathbf{p},\lambda)\mathrm{e}^{-\mathrm{i}p\cdot x}$$

6. **Majorana 费米子 χ 出射外线：**



$$\chi,\lambda = \langle \mathbf{p},\lambda|\,\bar{\chi}(x)\,|0\rangle = \bar{u}(\mathbf{p},\lambda)\mathrm{e}^{\mathrm{i}p\cdot x}$$



$$\chi,\lambda = \langle \mathbf{p},\lambda|\,\chi(x)\,|0\rangle = v(\mathbf{p},\lambda)\mathrm{e}^{\mathrm{i}p\cdot x}$$

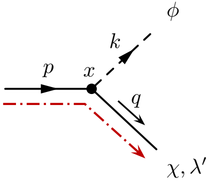
Majorana 费米子线上没有箭头，Feynman 规则依赖于**费米子流方向**与**动量方向**之间的异同

从每条连续费米子线写出散射振幅时，总是**逆着用点划线表示的费米子流方向**逐项写下费米子的贡献

第一种方法 Feynman 图

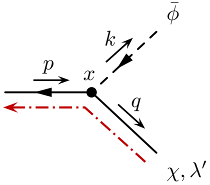
对于上述 $\psi \to \chi\phi$ 和 $\bar{\psi} \to \chi\bar{\phi}$ 过程，**第一种计算方法**对应于

$$\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \, \mathrm{i}T_1^{(1)} \, | \mathbf{p}^+, \lambda \rangle =$$



$$= \mathrm{i}\kappa \int \mathrm{d}^4x \, \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) \mathrm{e}^{-\mathrm{i}(p-q-k)\cdot x}$$

$$\langle \mathbf{q}, \lambda'; \mathbf{k}^- | \, \mathrm{i}T_2^{(1)} \, | \mathbf{p}^-, \lambda \rangle =$$

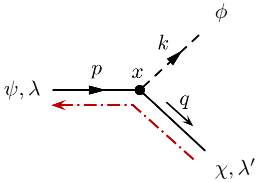


$$= -\mathrm{i}\kappa \int \mathrm{d}^4x \, \bar{v}(\mathbf{p}, \lambda) \Gamma_2 v(\mathbf{q}, \lambda') \mathrm{e}^{-\mathrm{i}(p-q-k)\cdot x}$$

第二种方法 Feynman 图

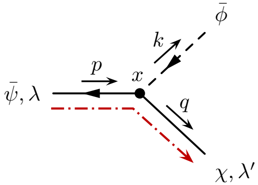
第二种计算方法对应于

$$\langle \mathbf{q},\lambda';\mathbf{k}^+|\,\mathrm{i}T_1^{(1)}\,|\mathbf{p}^+,\lambda\rangle = $$



$$= -\mathrm{i}\kappa\int \mathrm{d}^4x\,\bar{v}(\mathbf{p},\lambda)\Gamma_1^{\mathrm{C}}v(\mathbf{q},\lambda')\mathrm{e}^{-\mathrm{i}(p-q-k)\cdot x}$$

$$\langle \mathbf{q},\lambda';\mathbf{k}^-|\,\mathrm{i}T_2^{(1)}\,|\mathbf{p}^-,\lambda\rangle = $$

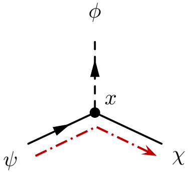


$$= \mathrm{i}\kappa\int \mathrm{d}^4x\,\bar{u}(\mathbf{q},\lambda')\Gamma_2^{\mathrm{C}}u(\mathbf{p},\lambda)\mathrm{e}^{-\mathrm{i}(p-q-k)\cdot x}$$

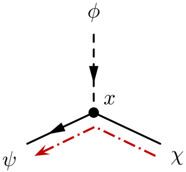
- **两种方法**在 Feynman 图上的**差异**只是**费米子流方向不同**，即**点划线箭头方向不同**
- 额外的**负号**来自两个**费米子场算符**的**交换**

位置空间顶点规则

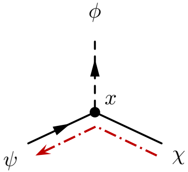
观察各个 Feynman 图元素与振幅表达式的关系，归纳出**位置空间**中的**顶点规则**



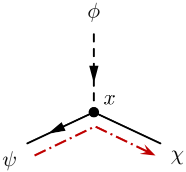
$$= i\kappa \int d^4x\, \Gamma_1 \,,$$



$$= i\kappa \int d^4x\, \Gamma_2$$



$$= i\kappa \int d^4x\, \Gamma_1^{\mathrm{C}} \,,$$

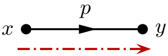


$$= i\kappa \int d^4x\, \Gamma_2^{\mathrm{C}}$$

Dirac 旋量场的 Feynman 传播子

研究 $\mathrm{i}T^{(2)}$ 的 T 矩阵元时可能遇到像 $\mathrm{N}[\bar{\chi}(y)\Gamma_1\overbracket{\psi(y)\bar{\psi}}(x)\Gamma_2\chi(x)]$ 这样的表达式

如果采用**第一种方法**进行计算，则 **Dirac 旋量场**的 **Feynman 传播子**在**位置空间**中的 Feynman 规则与 7.1.1 小节规则类似，表达为

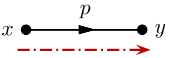


$$= \overbracket{\psi(y)\bar{\psi}}(x) = S_{\mathrm{F}}(y-x) = \int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{i}(\not{p}+m_\psi)}{p^2-m_\psi^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(y-x)}$$

Dirac 旋量场的 Feynman 传播子

研究 $\mathrm{i}T^{(2)}$ 的 T 矩阵元时可能遇到像 $\mathsf{N}[\bar\chi(y)\Gamma_1\overbracket{\psi(y)\bar\psi}(x)\Gamma_2\chi(x)]$ 这样的表达式

如果采用**第一种方法**进行计算，则 **Dirac 旋量场**的 **Feynman 传播子**在**位置空间**中的 Feynman 规则与 7.1.1 小节规则类似，表达为

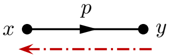


$$= \overbracket{\psi(y)\bar\psi}(x) = S_{\mathrm{F}}(y-x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(\not{p}+m_\psi)}{p^2-m_\psi^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(y-x)}$$

由 $\bar\chi\Gamma_1\psi = \bar\psi^{\mathrm{C}}\Gamma_1^{\mathrm{C}}\chi$ 和 $\bar\psi\Gamma_2\chi = \bar\chi\Gamma_2^{\mathrm{C}}\psi^{\mathrm{C}}$ 推出

$$\begin{aligned}\mathsf{N}[\bar\chi(y)\Gamma_1\overbracket{\psi(y)\bar\psi}(x)\Gamma_2\chi(x)] &= \mathsf{N}[\overbracket{\bar\psi^{\mathrm{C}}(y)\Gamma_1^{\mathrm{C}}\chi(y)\bar\chi(x)\Gamma_2^{\mathrm{C}}\psi^{\mathrm{C}}}(x)]\\ &= \mathsf{N}[\bar\chi(x)\Gamma_2^{\mathrm{C}}\overbracket{\psi^{\mathrm{C}}(x)\bar\psi^{\mathrm{C}}}(y)\Gamma_1^{\mathrm{C}}\chi(y)]\end{aligned}$$

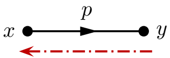
如果采用**第二种方法**进行计算，则相应的 **Feynman 传播子**是



$$= \overbracket{\psi^{\mathrm{C}}(x)\bar\psi^{\mathrm{C}}}(y) = \langle 0|\,\mathsf{T}[\psi^{\mathrm{C}}(x)\bar\psi^{\mathrm{C}}(y)]\,|0\rangle = \langle 0|\,\mathsf{T}[\mathcal{C}\bar\psi^{\mathrm{T}}(x)\psi^{\mathrm{T}}(y)\mathcal{C}]\,|0\rangle$$

Majorana 旋量场的 Feynman 传播子

进一步计算得到

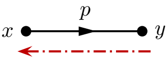


$$
\begin{aligned}
&= \psi^{\mathrm{C}}(x)\bar{\psi}^{\mathrm{C}}(y) = \langle 0|\,\mathsf{T}[\mathcal{C}\bar{\psi}^{\mathrm{T}}(x)\psi^{\mathrm{T}}(y)\mathcal{C}]\,|0\rangle \\
&= -\mathcal{C}\{\langle 0|\,\mathsf{T}[\psi(y)\bar{\psi}(x)]\,|0\rangle\}^{\mathrm{T}}\mathcal{C} = \mathcal{C}^{-1}[\psi(y)\bar{\psi}(x)]^{\mathrm{T}}\mathcal{C} \\
&= \mathcal{C}^{-1}S_{\mathrm{F}}^{\mathrm{T}}(y-x)\mathcal{C} = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\mathcal{C}^{-1}\mathrm{i}(\not{p}+m_\psi)^{\mathrm{T}}\mathcal{C}}{p^2-m_\psi^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(y-x)} \\
&= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\mathrm{i}(-\not{p}+m_\psi)}{p^2-m_\psi^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(y-x)}
\end{aligned}
$$

最后一步用到 $\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = -\gamma^\mu$

Majorana 旋量场的 Feynman 传播子

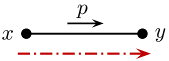
进一步计算得到



$$
\begin{aligned}
&= \psi^{\mathrm{C}}(x)\bar{\psi}^{\mathrm{C}}(y) = \langle 0|\,\mathsf{T}[\mathcal{C}\bar{\psi}^{\mathrm{T}}(x)\psi^{\mathrm{T}}(y)\mathcal{C}]\,|0\rangle \\
&= -\mathcal{C}\{\langle 0|\,\mathsf{T}[\psi(y)\bar{\psi}(x)]\,|0\rangle\}^{\mathrm{T}}\mathcal{C} = \mathcal{C}^{-1}[\psi(y)\bar{\psi}(x)]^{\mathrm{T}}\mathcal{C} \\
&= \mathcal{C}^{-1}S_{\mathrm{F}}^{\mathrm{T}}(y-x)\mathcal{C} = \int \frac{d^4p}{(2\pi)^4}\frac{\mathcal{C}^{-1}\mathrm{i}(\not{p}+m_\psi)^{\mathrm{T}}\mathcal{C}}{p^2-m_\psi^2+\mathrm{i}\epsilon}\,e^{-\mathrm{i}p\cdot(y-x)} \\
&= \int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{i}(-\not{p}+m_\psi)}{p^2-m_\psi^2+\mathrm{i}\epsilon}\,e^{-\mathrm{i}p\cdot(y-x)}
\end{aligned}
$$

最后一步用到 $\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = -\gamma^\mu$

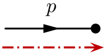
另一方面，**Majorana 旋量场**的 **Feynman 传播子**为

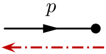


$$
= \chi(y)\bar{\chi}(x) = S_{\mathrm{F}}(y-x) = \int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{i}(\not{p}+m_\chi)}{p^2-m_\chi^2+\mathrm{i}\epsilon}\,e^{-\mathrm{i}p\cdot(y-x)}
$$

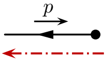
动量空间 Feynman 规则

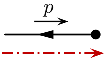
转换到**动量空间**，推出以下 **Feynman 规则**

1. **Dirac 正费米子 ψ 入射外线**： ψ, λ 

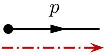
$= u(\mathbf{p}, \lambda)$， ψ, λ 

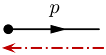
$= \bar{v}(\mathbf{p}, \lambda)$

2. **Dirac 反费米子 $\bar{\psi}$ 入射外线**： $\bar{\psi}, \lambda$ 

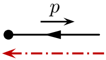
$= \bar{v}(\mathbf{p}, \lambda)$， $\bar{\psi}, \lambda$ 

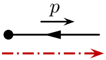
$= u(\mathbf{p}, \lambda)$

3. **Dirac 正费米子 ψ 出射外线**： 

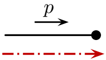
$\psi, \lambda = \bar{u}(\mathbf{p}, \lambda)$， 

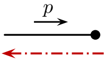
$\psi, \lambda = v(\mathbf{p}, \lambda)$

4. **Dirac 反费米子 $\bar{\psi}$ 出射外线**： 

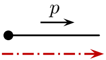
$\bar{\psi}, \lambda = v(\mathbf{p}, \lambda)$， 

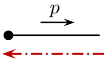
$\bar{\psi}, \lambda = \bar{u}(\mathbf{p}, \lambda)$

5. **Majorana 费米子 χ 入射外线**： χ, λ 

$= u(\mathbf{p}, \lambda)$， χ, λ 

$= \bar{v}(\mathbf{p}, \lambda)$

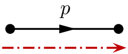
6. **Majorana 费米子 χ 出射外线**： 

$\chi, \lambda = \bar{u}(\mathbf{p}, \lambda)$， 

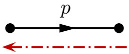
$\chi, \lambda = v(\mathbf{p}, \lambda)$

动量空间 Feynman 规则

7 **Dirac 费米子传播子**：

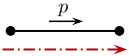


$$= \frac{\mathrm{i}(\not{p} + m_\psi)}{p^2 - m_\psi^2 + \mathrm{i}\epsilon}$$



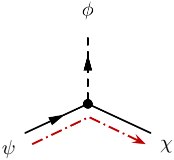
$$= \frac{\mathrm{i}(-\not{p} + m_\psi)}{p^2 - m_\psi^2 + \mathrm{i}\epsilon}$$

8 **Majorana 费米子传播子**：

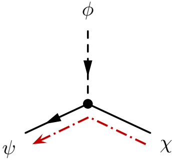


$$= \frac{\mathrm{i}(\not{p} + m_\chi)}{p^2 - m_\chi^2 + \mathrm{i}\epsilon}$$

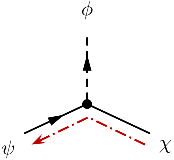
9 **Yukawa 相互作用顶点**：



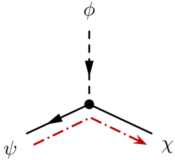
$= \mathrm{i}\kappa\,\Gamma_1,$



$= \mathrm{i}\kappa\,\Gamma_2$



$= \mathrm{i}\kappa\,\Gamma_1^{\mathrm{C}},$



$= \mathrm{i}\kappa\,\Gamma_2^{\mathrm{C}}$

Majorana 旋量场与对称性因子

🦉 注意，Majorana 费米子是纯中性粒子

🚗 如果末态包含超过 1 个全同的 Majorana 费米子

🚙 计算散射截面或衰变宽度时需要考虑末态对称性因子 $\mathcal{S}$

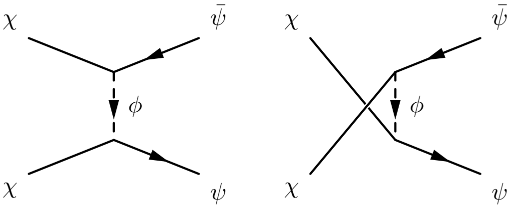
🛻 假如拉氏量的某个相互作用项包含 2 个或以上全同的 Majorana 旋量场

🚛 类似于 7.3 节的讨论，在导出顶点 Feynman 规则时需要考虑组合因子

🚚 计算时还需要留意 **Feynman 图的对称性因子**

9.7.3 小节　应用

- 下面应用上一小节推导出来的 **Feynman 规则**进行计算
- 考虑 $\chi\chi \to \psi\bar{\psi}$ **湮灭过程**
- 领头阶 Feynman 图如下图所示，包含一个 t **通道**和一个 u **通道**的 Feynman 图

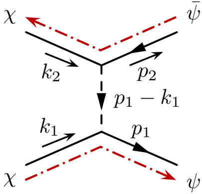


t 通道　　u 通道

- 现在，**费米子流方向**有**多种取法**，但各种取法的计算结果应该是**等价**的
- 在画出**拓扑不等价**的 Feynman 图时，**不需要**考虑费米子流的方向

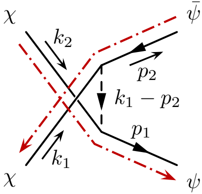
费米子流方向第一种取法

- 设初态两个 Majorana 费米子 χ 的四维动量为 k_1^μ 和 k_2^μ，末态 Dirac 费米子 ψ 和 $\bar{\psi}$ 的四维动量为 p_1^μ 和 p_2^μ，令 $t=(k_1-p_1)^2$，$u=(k_1-p_2)^2$
- 添加**带箭头的点划线**表示**费米子流方向**
- 应用动量空间 Feynman 规则，t **通道**和 u **通道** Feynman 图贡献的不变振幅是

$\mathrm{i}\mathcal{M}_t =$ 

$$= \bar{u}(p_1)(\mathrm{i}\kappa\Gamma_2)u(k_1)\,\frac{\mathrm{i}}{(p_1-k_1)^2-m_\phi^2}\,\bar{v}(k_2)(\mathrm{i}\kappa\Gamma_1)v(p_2)$$

$$= -\frac{\mathrm{i}\kappa^2}{t-m_\phi^2}\,\bar{u}(p_1)\Gamma_2 u(k_1)\bar{v}(k_2)\Gamma_1 v(p_2)$$

$\mathrm{i}\mathcal{M}_u =$ 

$$= \bar{v}(k_1)(\mathrm{i}\kappa\Gamma_1)v(p_2)\,\frac{\mathrm{i}}{(k_1-p_2)^2-m_\phi^2}\,\bar{u}(p_1)(\mathrm{i}\kappa\Gamma_2)u(k_2)$$

$$= -\frac{\mathrm{i}\kappa^2}{u-m_\phi^2}\,\bar{v}(k_1)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_2)$$

第一种取法的相对符号

根据

$$\langle \mathbf{p}_1^+;\, \mathbf{p}_2^- |\, \mathsf{N}[\phi(x)\bar{\psi}_a(x)(\Gamma_2)_{ab}\chi_b(x)\phi^\dagger(y)\bar{\chi}_c(y)(\Gamma_1)_{cd}\psi_d(y)]\, |\mathbf{k}_1;\, \mathbf{k}_2\rangle$$

$$+\langle \mathbf{p}_1^+;\, \mathbf{p}_2^- |\, \mathsf{N}[\phi(x)\bar{\psi}_a(x)(\Gamma_2)_{ab}\chi_b(x)\phi^\dagger(y)\bar{\chi}_c(y)(\Gamma_1)_{cd}\psi_d(y)]\, |\mathbf{k}_1;\, \mathbf{k}_2\rangle$$

$$=\langle \mathbf{p}_1^+;\, \mathbf{p}_2^- |\, \mathsf{N}[\psi_d(y)\bar{\psi}_a(x)(\Gamma_2)_{ab}\phi(x)\phi^\dagger(y)(\Gamma_1)_{cd}\bar{\chi}_c(y)\chi_b(x)]\, |\mathbf{k}_1;\, \mathbf{k}_2\rangle$$

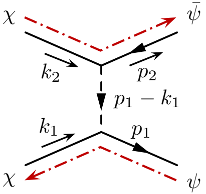
$$-\langle \mathbf{p}_1^+;\, \mathbf{p}_2^- |\, \mathsf{N}[\psi_d(y)\bar{\psi}_a(x)(\Gamma_2)_{ab}\phi(x)\phi^\dagger(y)(\Gamma_1)_{cd}\chi_b(x)\bar{\chi}_c(y)]\, |\mathbf{k}_1;\, \mathbf{k}_2\rangle$$

这两个 Feynman 图的相对符号为负

因而总振幅是 $i\mathcal{M} = i\mathcal{M}_t - i\mathcal{M}_u$

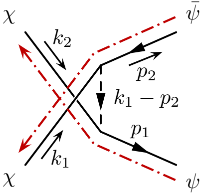
费米子流方向第二种取法

- 当然，也可以选择**其它费米子流方向**进行计算
- 比如，同时**反转**上述 t 通道 Feynman 图中**两条点划线的方向**，则 t **通道**振幅变成



$$i\tilde{\mathcal{M}}_t = \bar v(k_1)(i\kappa\Gamma_2^{C})v(p_1)\frac{i}{(p_1-k_1)^2-m_\phi^2}\bar u(p_2)(i\kappa\Gamma_1^{C})u(k_2)$$
$$= -\frac{i\kappa^2}{t-m_\phi^2}\bar v(k_1)\Gamma_2^{C}v(p_1)\bar u(p_2)\Gamma_1^{C}u(k_2)$$

- **反转**上述 u 通道 Feynman 图中**一条点划线的方向**，u **通道**振幅化为



$$i\tilde{\mathcal{M}}_u = \bar v(k_1)(i\kappa\Gamma_1)v(p_2)\frac{i}{(k_1-p_2)^2-m_\phi^2}\bar v(k_2)(i\kappa\Gamma_2^{C})v(p_1)$$
$$= -\frac{i\kappa^2}{u-m_\phi^2}\bar v(k_1)\Gamma_1 v(p_2)\bar v(k_2)\Gamma_2^{C}v(p_1)$$

第二种取法的相对符号

- 根据

$$
\begin{aligned}
&\langle \mathbf{p}_1^+;\, \mathbf{p}_2^- \,|\, \mathsf{N}[\phi(x)\bar{\chi}_a(x)(\Gamma_2^{\mathrm{C}})_{ab}\psi_b^{\mathrm{C}}(x)\phi^\dagger(y)\bar{\psi}_c^{\mathrm{C}}(y)(\Gamma_1^{\mathrm{C}})_{cd}\chi_d(y)]\,|\mathbf{k}_1;\, \mathbf{k}_2\rangle \\
&+ \langle \mathbf{p}_1^+;\, \mathbf{p}_2^- \,|\, \mathsf{N}[\phi(x)\bar{\chi}_a(x)(\Gamma_2^{\mathrm{C}})_{ab}\psi_b^{\mathrm{C}}(x)\phi^\dagger(y)\bar{\chi}_c(y)(\Gamma_1)_{cd}\psi_d(y)]\,|\mathbf{k}_1;\, \mathbf{k}_2\rangle \\
= &\langle \mathbf{p}_1^+;\, \mathbf{p}_2^- \,|\, \mathsf{N}[\bar{\psi}_c^{\mathrm{C}}(y)\psi_b^{\mathrm{C}}(x)(\Gamma_2^{\mathrm{C}})_{ab}\phi(x)\phi^\dagger(y)(\Gamma_1^{\mathrm{C}})_{cd}\chi_d(y)\bar{\chi}_a(x)]\,|\mathbf{k}_1;\, \mathbf{k}_2\rangle \\
&+ \langle \mathbf{p}_1^+;\, \mathbf{p}_2^- \,|\, \mathsf{N}[\psi_d(y)\psi_b^{\mathrm{C}}(x)(\Gamma_2^{\mathrm{C}})_{ab}\phi(x)\phi^\dagger(y)\bar{\chi}_a(x)\bar{\chi}_c(y)(\Gamma_1)_{cd}]\,|\mathbf{k}_1;\, \mathbf{k}_2\rangle
\end{aligned}
$$

- 这两个 Feynman 图的**相对符号**为**正**

- 因而**总振幅**是 $\mathrm{i}\tilde{\mathcal{M}} = \mathrm{i}\tilde{\mathcal{M}}_t + \mathrm{i}\tilde{\mathcal{M}}_u$

两种取法的等价性

$$
\begin{aligned}
\bar{v}(k_1)\Gamma_2^{\mathrm{C}}v(p_1)\bar{u}(p_2)\Gamma_1^{\mathrm{C}}u(k_2) &= u^{\mathrm{T}}(k_1)\,\mathcal{C}\mathcal{C}^{-1}\Gamma_2^{\mathrm{T}}\mathcal{C}\mathcal{C}\bar{u}^{\mathrm{T}}(p_1)v^{\mathrm{T}}(p_2)\,\mathcal{C}\mathcal{C}^{-1}\Gamma_1^{\mathrm{T}}\mathcal{C}\mathcal{C}\bar{v}^{\mathrm{T}}(k_2) \\
&= [u^{\mathrm{T}}(k_1)\Gamma_2^{\mathrm{T}}\bar{u}^{\mathrm{T}}(p_1)v^{\mathrm{T}}(p_2)\Gamma_1^{\mathrm{T}}\bar{v}^{\mathrm{T}}(k_2)]^{\mathrm{T}} \\
&= \bar{v}(k_2)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_1) \\
\bar{v}(k_1)\Gamma_1 v(p_2)\bar{v}(k_2)\Gamma_2^{\mathrm{C}}v(p_1) &= \bar{v}(k_1)\Gamma_1 v(p_2)u^{\mathrm{T}}(k_2)\,\mathcal{C}\mathcal{C}^{-1}\Gamma_2^{\mathrm{T}}\mathcal{C}\mathcal{C}\bar{u}^{\mathrm{T}}(p_1) \\
&= -\bar{v}(k_1)\Gamma_1 v(p_2)[u^{\mathrm{T}}(k_2)\,\Gamma_2^{\mathrm{T}}\bar{u}^{\mathrm{T}}(p_1)]^{\mathrm{T}} \\
&= -\bar{v}(k_1)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_2)
\end{aligned}
$$

➡️

$$
\begin{aligned}
\mathrm{i}\tilde{\mathcal{M}}_t &= -\frac{\mathrm{i}\kappa^2}{t-m_\phi^2}\,\bar{v}(k_1)\Gamma_2^{\mathrm{C}}v(p_1)\bar{u}(p_2)\Gamma_1^{\mathrm{C}}u(k_2) \\
&= -\frac{\mathrm{i}\kappa^2}{t-m_\phi^2}\,\bar{u}(p_1)\Gamma_2 u(k_1)\bar{v}(k_2)\Gamma_1 v(p_2) = \mathrm{i}\mathcal{M}_t \\
\mathrm{i}\tilde{\mathcal{M}}_u &= -\frac{\mathrm{i}\kappa^2}{u-m_\phi^2}\,\bar{v}(k_1)\Gamma_1 v(p_2)\bar{v}(k_2)\Gamma_2^{\mathrm{C}}v(p_1) \\
&= +\frac{\mathrm{i}\kappa^2}{u-m_\phi^2}\,\bar{v}(k_1)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_2) = -\mathrm{i}\mathcal{M}_u
\end{aligned}
$$

两种取法的等价性

$$\begin{aligned}\bar v(k_1)\Gamma_2^{\rm C}v(p_1)\bar u(p_2)\Gamma_1^{\rm C}u(k_2) &= u^{\rm T}(k_1)\,\mathcal{C}\mathcal{C}^{-1}\Gamma_2^{\rm T}\mathcal{C}\mathcal{C}\bar u^{\rm T}(p_1)v^{\rm T}(p_2)\,\mathcal{C}\mathcal{C}^{-1}\Gamma_1^{\rm T}\mathcal{C}\mathcal{C}\bar v^{\rm T}(k_2)\\ &= [u^{\rm T}(k_1)\Gamma_2^{\rm T}\bar u^{\rm T}(p_1)v^{\rm T}(p_2)\Gamma_1^{\rm T}\bar v^{\rm T}(k_2)]^{\rm T}\\ &= \bar v(k_2)\Gamma_1 v(p_2)\bar u(p_1)\Gamma_2 u(k_1)\\ \bar v(k_1)\Gamma_1 v(p_2)\bar v(k_2)\Gamma_2^{\rm C}v(p_1) &= \bar v(k_1)\Gamma_1 v(p_2)u^{\rm T}(k_2)\,\mathcal{C}\mathcal{C}^{-1}\Gamma_2^{\rm T}\mathcal{C}\mathcal{C}\bar u^{\rm T}(p_1)\\ &= -\bar v(k_1)\Gamma_1 v(p_2)[u^{\rm T}(k_2)\,\Gamma_2^{\rm T}\bar u^{\rm T}(p_1)]^{\rm T}\\ &= -\bar v(k_1)\Gamma_1 v(p_2)\bar u(p_1)\Gamma_2 u(k_2)\end{aligned}$$

➡️

$$\begin{aligned}{\rm i}\tilde{\mathcal{M}}_t &= -\frac{{\rm i}\kappa^2}{t-m_\phi^2}\,\bar v(k_1)\Gamma_2^{\rm C}v(p_1)\bar u(p_2)\Gamma_1^{\rm C}u(k_2)\\ &= -\frac{{\rm i}\kappa^2}{t-m_\phi^2}\,\bar u(p_1)\Gamma_2 u(k_1)\bar v(k_2)\Gamma_1 v(p_2) = {\rm i}\mathcal{M}_t\\ {\rm i}\tilde{\mathcal{M}}_u &= -\frac{{\rm i}\kappa^2}{u-m_\phi^2}\,\bar v(k_1)\Gamma_1 v(p_2)\bar v(k_2)\Gamma_2^{\rm C}v(p_1)\\ &= +\frac{{\rm i}\kappa^2}{u-m_\phi^2}\,\bar v(k_1)\Gamma_1 v(p_2)\bar u(p_1)\Gamma_2 u(k_2) = -{\rm i}\mathcal{M}_u\end{aligned}$$

🐧 可见，根据**费米子流方向**的**不同取法**计算出来的结果确实是**等价**的

🐣 因此 ${\rm i}\tilde{\mathcal{M}} = {\rm i}\tilde{\mathcal{M}}_t + {\rm i}\tilde{\mathcal{M}}_u = {\rm i}\mathcal{M}_t - {\rm i}\mathcal{M}_u = {\rm i}\mathcal{M}$

非极化振幅模方

接下来计算 $\chi\chi \to \psi\bar{\psi}$ 的**非极化振幅模方**

$$\overline{|\mathcal{M}|^2} = \overline{|\mathcal{M}_t - \mathcal{M}_u|^2} = \overline{|\mathcal{M}_t|^2} + \overline{|\mathcal{M}_u|^2} - (\overline{\mathcal{M}_t^* \mathcal{M}_u} + \text{H.c.})$$

使用具体形式 $\Gamma_1 = P_\text{R}$ 和 $\Gamma_2 = P_\text{L}$，由第一种取法的振幅计算结果得到

$$\text{i}\mathcal{M}_t = -\frac{\text{i}\kappa^2}{t - m_\phi^2}\,\bar{u}(p_1)P_\text{L}u(k_1)\bar{v}(k_2)P_\text{R}v(p_2)$$

$$(\text{i}\mathcal{M}_t)^* = \frac{\text{i}\kappa^2}{t - m_\phi^2}\,\bar{u}(k_1)P_\text{R}u(p_1)\bar{v}(p_2)P_\text{L}v(k_2)$$

$$\text{i}\mathcal{M}_u = -\frac{\text{i}\kappa^2}{u - m_\phi^2}\,\bar{v}(k_1)P_\text{R}v(p_2)\bar{u}(p_1)P_\text{L}u(k_2)$$

$$(\text{i}\mathcal{M}_u)^* = \frac{\text{i}\kappa^2}{u - m_\phi^2}\,\bar{v}(p_2)P_\text{L}v(k_1)\bar{u}(k_2)P_\text{R}u(p_1)$$

单纯 t 通道贡献

🐍 由 $P_L\gamma^\mu = \gamma^\mu P_R$、$P_R\gamma^\mu = \gamma^\mu P_L$、$P_L^2 = P_L$、$P_R^2 = P_R$ 和 $P_L P_R = P_R P_L = \mathbf{0}$ 得

$$\mathrm{tr}[(\not{p}_1 + m_\psi)P_L(\not{k}_1 + m_\chi)P_R] = \mathrm{tr}[(\not{p}_1 + m_\psi)(\not{k}_1 P_R + m_\chi P_L)P_R]$$

$$= \mathrm{tr}[(\not{p}_1 + m_\psi)\not{k}_1 P_R] = \frac{1}{2}\mathrm{tr}[(\not{p}_1 + m_\psi)\not{k}_1(1+\gamma^5)] = \frac{1}{2}\mathrm{tr}(\not{p}_1\not{k}_1) = 2\,k_1\cdot p_1$$

$$\mathrm{tr}[(\not{k}_2 - m_\chi)P_R(\not{p}_2 - m_\psi)P_L] = \frac{1}{2}\mathrm{tr}[(\not{k}_2 - m_\chi)\not{p}_2(1-\gamma^5)] = 2\,k_2\cdot p_2$$

✈️ 从而，**单纯 t 通道**对非极化振幅模方的贡献是

$$\overline{|\mathcal{M}_t|^2} = \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_t|^2$$

$$= \frac{\kappa^4}{4(t-m_\phi^2)^2}\sum_{\text{spins}}\bar{u}(p_1)P_L u(k_1)\bar{u}(k_1)P_R u(p_1)\bar{v}(k_2)P_R v(p_2)\bar{v}(p_2)P_L v(k_2)$$

$$= \frac{\kappa^4}{2\cdot 2(t-m_\phi^2)^2}\mathrm{tr}[(\not{p}_1 + m_\psi)P_L(\not{k}_1 + m_\chi)P_R]\mathrm{tr}[(\not{k}_2 - m_\chi)P_R(\not{p}_2 - m_\psi)P_L]$$

$$= \frac{\kappa^4(k_1\cdot p_1)(k_2\cdot p_2)}{(t-m_\phi^2)^2}$$

单纯 u 通道贡献和交叉贡献

🐢 另一方面，单纯 u 通道的贡献为

$$\overline{|\mathcal{M}_u|^2} = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_u|^2$$

$$= \frac{\kappa^4}{4(u-m_\phi^2)^2}\sum_{\text{spins}} \bar{v}(k_1)P_\text{R}v(p_2)\bar{v}(p_2)P_\text{L}v(k_1)\bar{u}(p_1)P_\text{L}u(k_2)\bar{u}(k_2)P_\text{R}u(p_1)$$

$$= \frac{\kappa^4}{2\cdot 2(u-m_\phi^2)^2}\text{tr}[(\not{k}_1 - m_\chi)P_\text{R}(\not{p}_2 - m_\psi)P_\text{L}]\text{tr}[(\not{p}_1 + m_\psi)P_\text{L}(\not{k}_2 + m_\chi)P_\text{R}]$$

$$= \frac{\kappa^4(k_1\cdot p_2)(k_2\cdot p_1)}{(u-m_\phi^2)^2}$$

🚀 而 t 和 u 通道的交叉贡献是

$$\overline{\mathcal{M}_t^*\mathcal{M}_u} = \frac{1}{4}\sum_{\text{spins}} \mathcal{M}_t^*\mathcal{M}_u$$

$$= \frac{\kappa^4}{4(t-m_\phi^2)(u-m_\phi^2)}\sum_{\text{spins}} \bar{u}(k_1)P_\text{R}u(p_1)\bar{u}(p_1)P_\text{L}u(k_2)\bar{v}(p_2)P_\text{L}v(k_2)\bar{v}(k_1)P_\text{R}v(p_2)$$

$\chi\chi \to \psi\bar{\psi}$ 非极化振幅模方

$$
\begin{aligned}
&\sum_{\text{spins}} \bar{u}(k_1)P_{\mathrm{R}}u(p_1)\bar{u}(p_1)P_{\mathrm{L}}u(k_2)\bar{v}(p_2)P_{\mathrm{L}}v(k_2)\bar{v}(k_1)P_{\mathrm{R}}v(p_2) \\
=& \sum_{\text{spins}} \bar{u}(k_1)P_{\mathrm{R}}u(p_1)\bar{u}(p_1)P_{\mathrm{L}}u(k_2)[u^{\mathrm{T}}(p_2)\,\mathcal{C}P_{\mathrm{L}}\mathcal{C}\bar{u}^{\mathrm{T}}(k_2)]^{\mathrm{T}}[u^{\mathrm{T}}(k_1)\,\mathcal{C}P_{\mathrm{R}}\mathcal{C}\bar{u}^{\mathrm{T}}(p_2)]^{\mathrm{T}} \\
=& \sum_{\text{spins}} \bar{u}(k_1)P_{\mathrm{R}}u(p_1)\bar{u}(p_1)P_{\mathrm{L}}u(k_2)\bar{u}(k_2)\,\mathcal{C}^{\mathrm{T}}P_{\mathrm{L}}^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}u(p_2)\bar{u}(p_2)\,\mathcal{C}^{\mathrm{T}}P_{\mathrm{R}}^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}u(k_1) \\
=& \ \mathrm{tr}[(\not{k}_1+m_\chi)P_{\mathrm{R}}(\not{p}_1+m_\psi)P_{\mathrm{L}}(\not{k}_2+m_\chi)\,\mathcal{C}^{-1}P_{\mathrm{L}}^{\mathrm{T}}\mathcal{C}(\not{p}_2+m_\psi)\,\mathcal{C}^{-1}P_{\mathrm{R}}^{\mathrm{T}}\mathcal{C}] \\
=& \ \mathrm{tr}[(\not{k}_1+m_\chi)\not{p}_1P_{\mathrm{L}}(\not{k}_2+m_\chi)P_{\mathrm{L}}(\not{p}_2+m_\psi)P_{\mathrm{R}}] = m_\chi\,\mathrm{tr}[(\not{k}_1+m_\chi)\not{p}_1P_{\mathrm{L}}(\not{p}_2+m_\psi)P_{\mathrm{R}}] \\
=& \ [(\not{k}_1+m_\chi)\not{p}_1\not{p}_2(1+\gamma^5)] = \frac{m_\chi^2}{2}\,\mathrm{tr}(\not{p}_1\not{p}_2) = 2m_\chi^2(p_1\cdot p_2)
\end{aligned}
$$

$$
\overline{\mathcal{M}_t^*\mathcal{M}_u} + \mathrm{H.c.} = \frac{\kappa^4 m_\chi^2(p_1\cdot p_2)}{2(t-m_\phi^2)(u-m_\phi^2)} + \mathrm{H.c.} = \frac{\kappa^4 m_\chi^2(p_1\cdot p_2)}{(t-m_\phi^2)(u-m_\phi^2)}
$$

于是，$\chi\chi \to \psi\bar{\psi}$ 的**非极化振幅模方**为

$$
\begin{aligned}
\overline{|\mathcal{M}|^2} &= \overline{|\mathcal{M}_t|^2} + \overline{|\mathcal{M}_u|^2} - (\overline{\mathcal{M}_t^*\mathcal{M}_u} + \mathrm{H.c.}) \\
&= \kappa^4\left[\frac{(k_1\cdot p_1)(k_2\cdot p_2)}{(t-m_\phi^2)^2} + \frac{(k_1\cdot p_2)(k_2\cdot p_1)}{(u-m_\phi^2)^2} - \frac{m_\chi^2(p_1\cdot p_2)}{(t-m_\phi^2)(u-m_\phi^2)}\right]
\end{aligned}
$$