

量子场论

第 6 章 量子场的相互作用

6.3 节和 6.4 节

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6.3 节 Wick 定理

 6.2 节借助时序乘积把 S 算符写成紧凑的 Dyson 级数形式

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \mathsf{T}[\mathcal{H}_1(x_1) \cdots \mathcal{H}_1(x_n)]$$

 如何适当地处理级数每一项中的时序乘积 $\mathsf{T}[\mathcal{H}_1(x_1) \cdots \mathcal{H}_1(x_n)]$ 呢？

 在量子场论中，相互作用哈密顿量密度 $\mathcal{H}_1(x)$ 是由若干个场算符构成的，因而需要处理的是多个场算符的时序乘积


 这不是一个简单问题，接下来将要介绍的 Wick 定理提供了一种简便的处理方法


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 在**相互作用绘景**中，**实标量场** $\phi(x)$ 的平面波展开式可以分解成两个部分

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

 **正能解**部分 $\phi^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ip \cdot x}$ 包含**湮灭算符**和 $e^{-ip \cdot x}$ **因子**

 **负能解**部分 $\phi^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger e^{ip \cdot x}$ 包含**产生算符**和 $e^{ip \cdot x}$ **因子**

矢量场和旋量场的正负能解


🐱 同样，可以把相互作用绘景中的有质量实矢量场 $A^\mu(x)$ 分解为

$$A^\mu(x) = A^{\mu(+)}(x) + A^{\mu(-)}(x)$$


🏈 正能解部分为 $A^{\mu(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm,0} \varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x}$


🏈 负能解部分为 $A^{\mu(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm,0} \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x}$


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 负能解部分为 $A^{\mu(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm,0} \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x}$

 相互作用绘景中 Dirac 旋量场 $\psi_a(x)$ 的平面波展开式也具有 Heisenberg 绘景中自由场展开式的形式，即

$$\psi_a(x) = \psi_a^{(+)}(x) + \psi_a^{(-)}(x)$$

 正能解部分为 $\psi_a^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm} u_a(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x}$

 负能解部分为 $\psi_a^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm} v_a(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x}$

 注意：各类量子场的正能解部分只包含湮灭算符，而负能解部分只包含产生算符

正规乘积

🐱 引入**正规乘积** (normal product) 的概念，以 N 为记号，它的作用是将乘积中的**所有湮灭算符**移动到**所有产生算符的右边**，形成**正规排序** (normal order)

🦾 正规排序对于**多个产生 (或湮灭) 算符之间谁先谁后**没有规定，每种排列都**等价**

🔪 移动算符的过程中如果涉及**奇数次费米子算符**之间的**交换**，就应该额外添加一个**负号**，这个规定与费米子产生湮灭算符的**反对易性**相匹配

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🔪 对于**标量场**的产生湮灭算符, 有 $N(a_p a_q^\dagger a_k a_l^\dagger) = a_q^\dagger a_l^\dagger a_p a_k = a_l^\dagger a_q^\dagger a_p a_k = a_l^\dagger a_q^\dagger a_k a_p$


🔪 第二步写出两个产生算符的另一种次序, 第三步写出两个湮灭算符的另一种次序, 而**对易关系** $[a_q^\dagger, a_l^\dagger] = [a_p, a_k] = 0$ 保证这些表达式是**等价的**


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 对于**旋量场**的产生湮灭算符, 有 $N(b_{p,\lambda_1} a_{q,\lambda_2}^\dagger a_{k,\lambda_3} b_{l,\lambda_4}^\dagger) = -a_{q,\lambda_2}^\dagger b_{l,\lambda_4}^\dagger b_{p,\lambda_1} a_{k,\lambda_3} = b_{l,\lambda_4}^\dagger a_{q,\lambda_2}^\dagger b_{p,\lambda_1} a_{k,\lambda_3} = -b_{l,\lambda_4}^\dagger a_{q,\lambda_2}^\dagger a_{k,\lambda_3} b_{p,\lambda_1}$

 第二步写出两个产生算符的另一种次序, 与第一步相差一个**负号**

 第三步写出两个湮灭算符的另一种次序, 与第二步相差一个**负号**


 **反对易关系** $\{a_{q,\lambda_2}^\dagger, b_{l,\lambda_4}^\dagger\} = \{b_{p,\lambda_1}, a_{k,\lambda_3}\} = 0$ 保证这些表达式是**等价的**

正规乘积的真空期待值

 正能解包含湮灭算符，负能解包含产生算符，正规排序中正能解处于负能解右边


 两个标量场算符的正规乘积为

$$N[\phi(x)\phi(y)] = \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x)$$

 最后一项中 $\phi^{(+)}(x)$ 被正规排序移动到 $\phi^{(-)}(y)$ 的右边

 两个旋量场算符的正规乘积为

$$N[\psi_a(x)\psi_b(y)] = \psi_a^{(-)}(x)\psi_b^{(-)}(y) + \psi_a^{(-)}(x)\psi_b^{(+)}(y) + \psi_a^{(+)}(x)\psi_b^{(+)}(y) - \psi_b^{(-)}(y)\psi_a^{(+)}(x)$$


 最后一项中 $\psi_a^{(+)}(x)$ 被正规排序移动到 $\psi_b^{(-)}(y)$ 的右边，并出现一个负号

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$$N[\phi(x)\phi(y)] = \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x)$$

 最后一项中 $\phi^{(+)}(x)$ 被正规排序移动到 $\phi^{(-)}(y)$ 的**右边**

 两个**旋量场算符**的正规乘积为

$$N[\psi_a(x)\psi_b(y)] = \psi_a^{(-)}(x)\psi_b^{(-)}(y) + \psi_a^{(-)}(x)\psi_b^{(+)}(y) + \psi_a^{(+)}(x)\psi_b^{(+)}(y) - \psi_b^{(-)}(y)\psi_a^{(+)}(x)$$

 最后一项中 $\psi_a^{(+)}(x)$ 被正规排序移动到 $\psi_b^{(-)}(y)$ 的**右边**，并出现一个**负号**

 湮灭算符对真空态 $|0\rangle$ 的作用为零，如 $a_p|0\rangle = 0$ ，相应地， $\langle 0|a_p^\dagger = 0$

 因此，对一组产生湮灭算符的**任意乘积取正规排序**之后，**真空期待值为零**，即

$$\langle 0| N(\text{产生湮灭算符的乘积}) |0\rangle = 0$$

一般场算符的正负能解


👾 用统一记号 $\Phi_a(x)$ 代表一般场算符，它可以是标量场 $\phi(x)$ 或 $\phi^\dagger(x)$ ，也可以是矢量场 $A^\mu(x)$ 的一个分量，还可以是旋量场 $\psi_a(x)$ 或 $\bar{\psi}_a(x)$ 的一个分量


🌀 比如， $\Phi_a(x)\Phi_b(x)\Phi_c(x)$ 可以表示 $\phi(x)\phi(x)\phi(x)$ ，也可以表示 $A_\mu(x)\bar{\psi}_a(x)\psi_b(x)$


🚗 后者不是 Lorentz 不变的，但可以利用 Dirac 矩阵线性地组合出 Lorentz 不变量


$$A_\mu(x)\bar{\psi}_a(x)(\gamma^\mu)_{ab}\psi_b(x) = A_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)$$

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 令 $\Phi_a(x) = \Phi_a^{(+)}(x) + \Phi_a^{(-)}(x)$ ，分解为正能解 $\Phi_a^{(+)}(x)$ 和负能解 $\Phi_a^{(-)}(x)$ ，则


$$\Phi_a(x)\Phi_b(y) = \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(-)}(y)$$

 由于正能解包含湮灭算符，负能解包含产生算符，有


$$\Phi_a^{(+)}(x)|0\rangle = 0, \quad \langle 0|\Phi_a^{(-)}(x) = 0$$

 从而推出


$$\langle 0|\Phi_a(x)\Phi_b(y)|0\rangle = \langle 0|\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)|0\rangle$$

 只有不满足正规排序的项 $\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)$ 才可能对真空期待值具有非零贡献

两个一般场算符的正规乘积


 将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的正规乘积分解为

$$\begin{aligned} N[\Phi_a(x)\Phi_b(y)] = & \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) \\ & + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) \end{aligned}$$


 因子 $\epsilon_{ab} = \pm 1$ 考虑了费米子算符的反对易性

 若 $\Phi_a(x)$ 和 $\Phi_b(y)$ 都是费米子算符，则 $\epsilon_{ab} = -1$ ；其余情况 $\epsilon_{ab} = +1$


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 将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的正规乘积分解为

$$\begin{aligned} N[\Phi_a(x)\Phi_b(y)] &= \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) \\ &\quad + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) \end{aligned}$$

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 若 $\Phi_a(x)$ 和 $\Phi_b(y)$ 都是费米子算符，则 $\epsilon_{ab} = -1$ ；其余情况 $\epsilon_{ab} = +1$

 利用 ϵ_{ab} 记号，依照两个产生(湮灭)算符的对易性或反对易性交换等式右边第一项(第三项)的两个场算符，得到

$$\begin{aligned} N[\Phi_a(x)\Phi_b(y)] &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \epsilon_{ab}\Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) \\ &= \epsilon_{ab}[\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \epsilon_{ab}\Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x)] \end{aligned}$$

即

$$N[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab} N[\Phi_b(y)\Phi_a(x)]$$

 交换两个场算符之后，正规乘积只相差一个由费米子算符反对易性导致的符号

两个一般场算符的时序乘积

🐎 另一方面，将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的**时序乘积**化为

$$\begin{aligned} T[\Phi_a(x)\Phi_b(y)] &= \Phi_a(x)\Phi_b(y)\theta(x^0 - y^0) + \epsilon_{ab}\Phi_b(y)\Phi_a(x)\theta(y^0 - x^0) \\ &= \epsilon_{ab}[\epsilon_{ab}\Phi_a(x)\Phi_b(y)\theta(x^0 - y^0) + \Phi_b(y)\Phi_a(x)\theta(y^0 - x^0)] \end{aligned}$$

🖨️ 交换两个场算符后，**时序乘积**也只相差一个由费米子算符的**反对易性**导致的**符号**

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🐎 另一方面，将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的**时序乘积**化为

$$\begin{aligned} \mathbb{T}[\Phi_a(x)\Phi_b(y)] &= \Phi_a(x)\Phi_b(y)\theta(x^0 - y^0) + \epsilon_{ab}\Phi_b(y)\Phi_a(x)\theta(y^0 - x^0) \\ &= \epsilon_{ab}[\epsilon_{ab}\Phi_a(x)\Phi_b(y)\theta(x^0 - y^0) + \Phi_b(y)\Phi_a(x)\theta(y^0 - x^0)] \end{aligned}$$

🚆 交换两个场算符后，**时序乘积**也只相差一个由费米子算符的**反对易性**导致的**符号**

$$\mathbb{T}[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab} \mathbb{T}[\Phi_b(y)\Phi_a(x)]$$

🚆 当 $x^0 \geq y^0$ 时， $\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积为

$$\begin{aligned} \mathbb{T}[\Phi_a(x)\Phi_b(y)] &= \Phi_a(x)\Phi_b(y) \\ &= \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(-)}(y) \end{aligned}$$

🚆 把最后一项改写成

$$\begin{aligned} \Phi_a^{(+)}(x)\Phi_b^{(-)}(y) &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_a^{(+)}(x)\Phi_b^{(-)}(y) - \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) \\ &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp} \end{aligned}$$

🚆 这里 $[\cdot, \cdot]_{-} = [\cdot, \cdot]$ 代表**对易子**， $[\cdot, \cdot]_{+} = \{\cdot, \cdot\}$ 代表**反对易子**

🚆 \mp 号仅当 $\Phi_a(x)$ 和 $\Phi_b(y)$ **都是费米子算符**时取**正号**，其余情况取**负号**

$x^0 \geq y^0$ 时的 $T[\Phi_a(x)\Phi_b(y)]$

🐏 于是推出

$$\begin{aligned}
T[\Phi_a(x)\Phi_b(y)] &= \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) \\
&\quad + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp} \\
&= N[\Phi_a(x)\Phi_b(y)] + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}
\end{aligned}$$

🏠 $[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}$ 必定是一个 **c 数**，因为 $\Phi_a^{(+)}(x)$ 中湮灭算符与 $\Phi_b^{(-)}(y)$ 中产生算符的对易子或反对易子并不是算符，而是 c 数

$x^0 \geq y^0$ 时的 $T[\Phi_a(x)\Phi_b(y)]$

🐑 于是推出

$$\begin{aligned}
T[\Phi_a(x)\Phi_b(y)] &= \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) \\
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&= N[\Phi_a(x)\Phi_b(y)] + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}
\end{aligned}$$


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🚆 由 $\Phi_a^{(+)}(x)|0\rangle = 0$ 和 $\langle 0|\Phi_a(x)\Phi_b(y)|0\rangle = \langle 0|\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)|0\rangle$ 得


$$\begin{aligned}
[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp} &= \langle 0|[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}|0\rangle = \langle 0|\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)|0\rangle \\
&= \langle 0|\Phi_a(x)\Phi_b(y)|0\rangle = \langle 0|T[\Phi_a(x)\Phi_b(y)]|0\rangle
\end{aligned}$$

🚆 因此当 $x^0 \geq y^0$ 时，有 $T[\Phi_a(x)\Phi_b(y)] = N[\Phi_a(x)\Phi_b(y)] + \langle 0|T[\Phi_a(x)\Phi_b(y)]|0\rangle$

$x^0 < y^0$ 时的 $T[\Phi_a(x)\Phi_b(y)]$

 当 $x^0 < y^0$ 时, $\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积变成

$$\begin{aligned} T[\Phi_a(x)\Phi_b(y)] &= \epsilon_{ab}\Phi_b(y)\Phi_a(x) \\ &= \epsilon_{ab}[\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(-)}(x)] \\ &= \epsilon_{ab}\{\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) \\ &\quad + \epsilon_{ab}\Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + [\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}\} \\ &= \epsilon_{ab} N[\Phi_b(y)\Phi_a(x)] + \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp} \\ &= N[\Phi_a(x)\Phi_b(y)] + \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp} \end{aligned}$$

 注意到

$$\begin{aligned} \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp} &= \epsilon_{ab} \langle 0 | [\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp} | 0 \rangle = \epsilon_{ab} \langle 0 | \Phi_b^{(+)}(y)\Phi_a^{(-)}(x) | 0 \rangle \\ &= \epsilon_{ab} \langle 0 | \Phi_b(y)\Phi_a(x) | 0 \rangle = \epsilon_{ab} \langle 0 | T[\Phi_b(y)\Phi_a(x)] | 0 \rangle \\ &= \langle 0 | T[\Phi_a(x)\Phi_b(y)] | 0 \rangle \end{aligned}$$

 因此也有 $T[\Phi_a(x)\Phi_b(y)] = N[\Phi_a(x)\Phi_b(y)] + \langle 0 | T[\Phi_a(x)\Phi_b(y)] | 0 \rangle$

场算符的缩并

🐧 于是，无论 x^0 和 y^0 孰大孰小， $\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积都可以统一地表达为

$$T[\Phi_a(x)\Phi_b(y)] = N[\Phi_a(x)\Phi_b(y)] + \langle 0| T[\Phi_a(x)\Phi_b(y)] |0\rangle$$

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🚗 引入场算符的缩并 (contraction) 概念，将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的缩并定义为

$$\overbrace{\Phi_a(x)\Phi_b(y)} \equiv \langle 0| T[\Phi_a(x)\Phi_b(y)] |0\rangle = \begin{cases} [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}, & x^0 \geq y^0 \\ \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}, & x^0 < y^0 \end{cases}$$

🚗 上式仅当 $\Phi_a^{(+)}(x)$ 中的湮灭算符与 $\Phi_b^{(-)}(y)$ 中的产生算符属于同一套产生湮灭算符时才能得到非零结果

🚗 不同类型的场算符的缩并为零，如 $\overbrace{\phi(x)A^\mu(y)} = \overbrace{\phi(x)\psi_a(y)} = \overbrace{A^\mu(x)\psi_a(y)} = 0$

🚗 两个场算符的缩并是一个 c 数，不会受到正规排序 N 的影响

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🚗 两个场算符的缩并是一个 c 数，不会受到正规排序 N 的影响

🚗 在正规乘积中出现缩并记号时，参与缩并的一对场算符可以不相邻

🚗 为了使它们相邻，需要适当地交换场算符，并考虑交换费米子算符时引起的符号差异，约定这样得到的式子与原先的式子相等，例如

$$N(\overbrace{\Phi_a\Phi_b\Phi_c\Phi_d\Phi_e\Phi_f}) = \epsilon_{cd}\epsilon_{ef}N(\overbrace{\Phi_a\Phi_b\Phi_d\Phi_c\Phi_f\Phi_e}) = \epsilon_{cd}\epsilon_{ef}\overbrace{\Phi_b\Phi_d\Phi_c\Phi_f}\overbrace{\Phi_a}\overbrace{\Phi_e}N(\Phi_a\Phi_e)$$

Wick 定理

🐶 将 $T[\Phi_a(x)\Phi_b(y)] = N[\Phi_a(x)\Phi_b(y)] + \langle 0|T[\Phi_a(x)\Phi_b(y)]|0\rangle$

改写为

$$T[\Phi_a(x)\Phi_b(y)] = N[\Phi_a(x)\Phi_b(y) + \overbrace{\Phi_a(x)\Phi_b(y)}]$$

🚒 可见，两个场算符的**时序乘积**等于它们的**正规乘积**加上它们的**缩并**

Wick 定理

🐶 将 $T[\Phi_a(x)\Phi_b(y)] = N[\Phi_a(x)\Phi_b(y)] + \langle 0|T[\Phi_a(x)\Phi_b(y)]|0\rangle$
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🚒 可见，两个场算符的**时序乘积**等于它们的**正规乘积**加上它们的**缩并**

🚌 这个结论可以**推广**成



Gian Wick (1909-1992)

Wick 定理

🚌 一组场算符的**时序乘积**可以分解为它们的**正规乘积**及**所有可能缩并**的正规乘积之和，也就是说，

$$T[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)] = N[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n) + (\Phi_{a_1}\Phi_{a_2}\cdots\Phi_{a_n} \text{ 的所有可能缩并)]$$


🚌 Wick 定理的**证明**见讲义 **6.3.2 小节**选读内容

Wick 定理应用举例

 Wick 定理中的缩并是在保持各个场算符位置的情况下直接添加缩并线

 对于四个场算符的情况，有


$$\begin{aligned}
 T(\Phi_a \Phi_b \Phi_c \Phi_d) = & N(\Phi_a \Phi_b \Phi_c \Phi_d + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} \\
 & + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} \\
 & + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d})
 \end{aligned}$$

 由于 $\langle 0 | N(\text{产生湮灭算符的乘积}) | 0 \rangle = 0$ ，上式的真空期待值为


$$\begin{aligned}
 & \langle 0 | T(\Phi_a \Phi_b \Phi_c \Phi_d) | 0 \rangle \\
 = & \langle 0 | N(\overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d}) | 0 \rangle \\
 = & N(\overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \epsilon_{bc} \overbrace{\Phi_a \Phi_c \Phi_b \Phi_d} + \epsilon_{cd} \epsilon_{bd} \overbrace{\Phi_a \Phi_d \Phi_b \Phi_c}) \\
 = & \langle 0 | T(\Phi_a \Phi_b) | 0 \rangle \langle 0 | T(\Phi_c \Phi_d) | 0 \rangle + \epsilon_{bc} \langle 0 | T(\Phi_a \Phi_c) | 0 \rangle \langle 0 | T(\Phi_b \Phi_d) | 0 \rangle \\
 & + \epsilon_{cd} \epsilon_{bd} \langle 0 | T(\Phi_a \Phi_d) | 0 \rangle \langle 0 | T(\Phi_b \Phi_c) | 0 \rangle
 \end{aligned}$$

6.4 节 Feynman 传播子

 在应用 **Wick 定理** 时，**两个场算符的缩并** 是一种基本要素

 上一节已经指出，仅当参与缩并的场算符中含有 **同一套产生湮灭算符** 时，缩并的结果才 **不为零**

 **Feynman 传播子** (propagator) 就是这样的 **非零缩并**

 本节将导出 Feynman 传播子的 **显式结果**



Richard Feynman
(1918–1988)

6.4.1 小节 实标量场的 Feynman 传播子

 实标量场 $\phi(x)$ 的 Feynman 传播子 $D_F(x-y)$ 定义为


$$D_F(x-y) \equiv \overbrace{\phi(x)\phi(y)} = \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$$

 根据正负能解展开式, 当 $x^0 > y^0$ 时, 有

$$\begin{aligned} \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle &= \langle 0 | \phi(x)\phi(y) | 0 \rangle = \langle 0 | \phi^{(+)}(x)\phi^{(-)}(y) | 0 \rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{4E_p E_q}} \langle 0 | a_p e^{-ip \cdot x} a_q^\dagger e^{iq \cdot y} | 0 \rangle = \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{4E_p E_q}} \langle 0 | [a_p, a_q^\dagger] | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^3 \sqrt{4E_p E_q}} \delta^{(3)}(\mathbf{p} - \mathbf{q}) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip \cdot (x-y)}}{2E_p} \\ &= \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot (x-y)} \frac{e^{-iE_p(x^0-y^0)}}{2E_p} \end{aligned}$$

 第四步用到产生湮灭算符的对易关系

! 注意, 上式中 $p^0 = E_p$, $q^0 = E_q$

 借助复变函数的知识, 可以将最后一行中的因子 $\frac{e^{-iE_p(x^0-y^0)}}{2E_p}$ 化为一维积分

复平面上的曲线积分

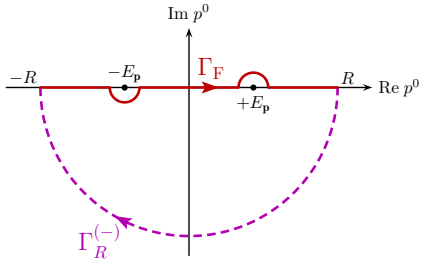
🍷 将 p^0 视作复变量，在 p^0 的复平面上考虑函数 $\frac{e^{-ip^0(x^0 - y^0)}}{(p^0 - E_p)(p^0 + E_p)}$ 的曲线积分

🏠 这个函数具有两个一阶极点， $p^0 = \pm E_p$ ，均位于实轴上

🍷 右下图画出 p^0 复平面上的两条积分路径

🏠 路径 Γ_F 在两个极点处分别通过一个半径无穷小的半圆绕过极点，当 $R \rightarrow \infty$ 时， Γ_F 将从 $p^0 = -\infty$ 一直延伸到 $p^0 = +\infty$

🍷 将 Γ_F 与下半平面上的半圆弧 $\Gamma_R^{(-)}$ 组成围线 $C_F^{(-)} = \Gamma_F + \Gamma_R^{(-)}$ ，曲线方向沿顺时针方向，即反方向



复平面上的曲线积分

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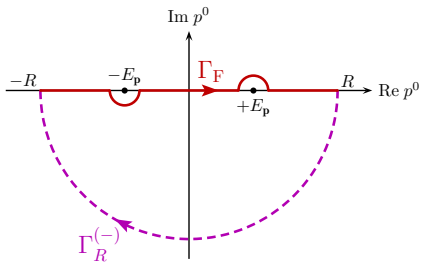
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🍷 由于 $x^0 - y^0 > 0$ ， $e^{-ip^0(x^0-y^0)}$ 中的因子 $e^{i\text{Im}(p^0)(x^0-y^0)}$ 在下半平面 ($\text{Im} p^0 < 0$) 随着 R 增大而指数衰减

🍷 根据复分析的 Jordan 引理，有

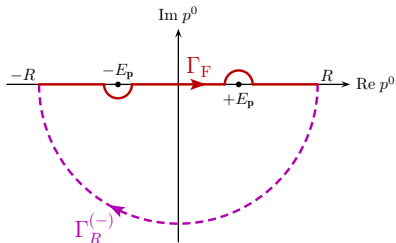
$$\lim_{R \rightarrow \infty} \int_{\Gamma_R^{(-)}} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} = 0$$



沿 Γ_F 的积分

🍔 当 $R \rightarrow \infty$ 时，通过留数定理计算相应的积分主值，得

$$\begin{aligned} \int_{\Gamma_F} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} &= \int_{C_F^{(-)} = \Gamma_F + \Gamma_R^{(-)}} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} \\ &= -2\pi i \operatorname{Res}_{p^0=E_p} \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} = -2\pi i \frac{e^{-iE_p(x^0-y^0)}}{2E_p} \end{aligned}$$



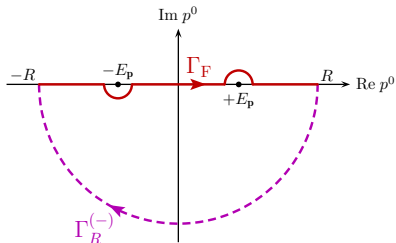
沿 Γ_F 的积分

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$$\begin{aligned} \int_{\Gamma_F} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_p)(p^0+E_p)} &= \int_{C_F^{(-)}=\Gamma_F+\Gamma_R^{(-)}} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_p)(p^0+E_p)} \\ &= -2\pi i \operatorname{Res}_{p^0=E_p} \frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_p)(p^0+E_p)} = -2\pi i \frac{e^{-iE_p(x^0-y^0)}}{2E_p} \end{aligned}$$

🍌 利用

$$\begin{aligned} (p^0-E_p)(p^0+E_p) &= (p^0)^2 - E_p^2 \\ &= (p^0)^2 - |\mathbf{p}|^2 - m^2 \\ &= p^2 - m^2 \end{aligned}$$



🍌 进一步得到

$$\frac{e^{-iE_p(x^0-y^0)}}{2E_p} = -\frac{1}{2\pi i} \int_{\Gamma_F} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_p)(p^0+E_p)} = \int_{\Gamma_F} \frac{dp^0}{2\pi} i \frac{e^{-ip^0(x^0-y^0)}}{p^2 - m^2}$$

沿实轴积分

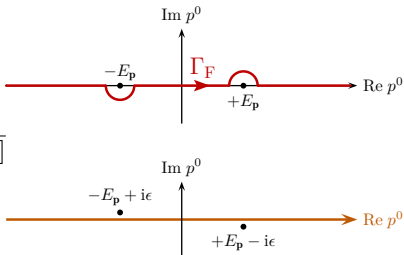
☪ 如右下图所示，如果将左边极点沿正虚轴方向移动一个**无穷小量** $\epsilon > 0$ ，右边极点沿负虚轴方向同样移动无穷小量 ϵ ，则**沿实轴积分**将等价于原来**沿 Γ_F 积分**

☪ 此时**极点**位置为 $p^0 = \pm(E_p - i\epsilon)$ ，被积函数中的**分母** $p^2 - m^2$ 应改成


$$\begin{aligned} [p^0 - (E_p - i\epsilon)][p^0 + (E_p - i\epsilon)] &= (p^0)^2 - (E_p - i\epsilon)^2 \\ &= (p^0)^2 - E_p^2 + 2i\epsilon E_p + \epsilon^2 \simeq p^2 - m^2 + i\epsilon \end{aligned}$$

☪ 最后一步忽略了 ϵ 的**二阶小量**，而 $2\epsilon E_p > 0$ 作为无穷小量**等价于** ϵ ，于是得到

$$\begin{aligned} \frac{e^{-iE_p(x^0 - y^0)}}{2E_p} &= \int_{\Gamma_F} \frac{dp^0}{2\pi} \frac{i e^{-ip^0(x^0 - y^0)}}{p^2 - m^2} \\ &= \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{i e^{-ip^0(x^0 - y^0)}}{[p^0 - (E_p - i\epsilon)][p^0 + (E_p - i\epsilon)]} \\ &= \int \frac{dp^0}{2\pi} \frac{i e^{-ip^0(x^0 - y^0)}}{p^2 - m^2 + i\epsilon} \end{aligned}$$



$x^0 > y^0$ 时的 $\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$

 将以上 $\frac{e^{-iE_p(x^0-y^0)}}{2E_p} = \int \frac{dp^0}{2\pi} \frac{i e^{-ip^0(x^0-y^0)}}{p^2 - m^2 + i\epsilon}$

代入 $x^0 > y^0$ 时 $\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$ 的表达式, 立即推出

$$\begin{aligned} \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \frac{e^{-iE_p(x^0-y^0)}}{2E_p} \\ &= \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{i e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{-ip^0(x^0-y^0)}}{p^2 - m^2 + i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \end{aligned}$$

$x^0 < y^0$ 的情况

🔧 当 $x^0 < y^0$ 时，时间排序将改变 $\phi(x)$ 和 $\phi(y)$ 的次序，有

$$\begin{aligned}\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle &= \langle 0 | \phi(y)\phi(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip \cdot (y-x)}}{2E_p} = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip \cdot (x-y)}}{2E_p} \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-ip \cdot (x-y)} \frac{e^{iE_p(x^0-y^0)}}{2E_p} = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot (x-y)} \frac{e^{iE_p(x^0-y^0)}}{2E_p}\end{aligned}$$

🍎 最后一步把积分变量 \mathbf{p} 替换成 $-\mathbf{p}$

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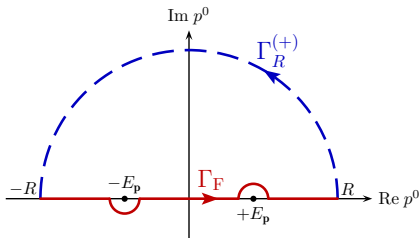
$$\begin{aligned} \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle &= \langle 0 | \phi(y)\phi(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip \cdot (y-x)}}{2E_p} = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip \cdot (x-y)}}{2E_p} \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-ip \cdot (x-y)} \frac{e^{iE_p(x^0-y^0)}}{2E_p} = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot (x-y)} \frac{e^{iE_p(x^0-y^0)}}{2E_p} \end{aligned}$$

🍎 最后一步把积分变量 p 替换成 $-p$

🌀 将 Γ_F 与上半平面上的半圆弧 $\Gamma_R^{(+)}$ 组成围线 $C_F^{(+)} = \Gamma_F + \Gamma_R^{(+)}$ ，曲线方向沿逆时针方向，即正方向

🍌 由于 $x^0 - y^0 < 0$ ， $e^{-ip^0(x^0-y^0)}$ 中的因子 $e^{\text{Im}(p^0)(x^0-y^0)}$ 在上半平面 ($\text{Im} p^0 > 0$) 随着 R 增大而指数衰减，由 Jordan 引理得

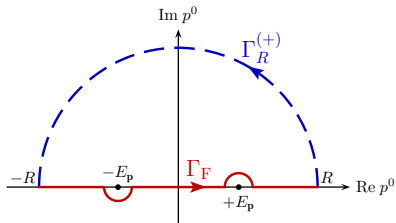
$$\lim_{R \rightarrow \infty} \int_{\Gamma_R^{(+)}} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} = 0$$



$x^0 < y^0$ 时的 $\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$

从而在 $R \rightarrow \infty$ 时推出

$$\begin{aligned} \int_{\Gamma_F} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} &= \int_{C_F^{(+)} = \Gamma_F + \Gamma_R^{(+)}} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} \\ &= 2\pi i \operatorname{Res}_{p^0 = -E_p} \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} = -2\pi i \frac{e^{iE_p(x^0-y^0)}}{2E_p} \end{aligned}$$



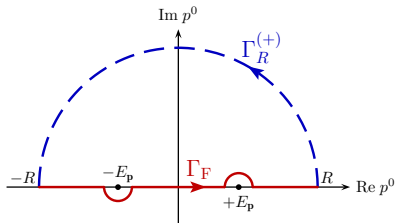
$x^0 < y^0$ 时的 $\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$

🍩 从而在 $R \rightarrow \infty$ 时推出

$$\begin{aligned} \int_{\Gamma_F} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} &= \int_{C_F^{(+)} = \Gamma_F + \Gamma_R^{(+)}} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} \\ &= 2\pi i \operatorname{Res}_{p^0 = -E_p} \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} = -2\pi i \frac{e^{iE_p(x^0-y^0)}}{2E_p} \end{aligned}$$

🍪 故

$$\begin{aligned} \frac{e^{iE_p(x^0-y^0)}}{2E_p} &= -\frac{1}{2\pi i} \int_{\Gamma_F} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{p^2 - m^2} \\ &= \int \frac{dp^0}{2\pi} \frac{i e^{-ip^0(x^0-y^0)}}{p^2 - m^2 + i\epsilon} \end{aligned}$$



🍪 代入 $x^0 < y^0$ 时 $\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$ 的表达式, 即得

$$\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot (x-y)} \frac{e^{iE_p(x^0-y^0)}}{2E_p} = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}$$

实标量场的 Feynman 传播子

🍰 可见，无论 $x^0 > y^0$ 还是 $x^0 < y^0$ ， $\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$ 的表达式都是一样的

🍦 因此，实标量场的 Feynman 传播子总可以表达为

$$D_F(x - y) = \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x - y)}$$

🍲 上式对 p^μ 的所有取值进行积分，因而 p^0 不一定等于 E_p

🍦 p^μ 不一定满足质壳条件 $p^2 = m^2$ ，满足质壳条件的 p^μ 对应于积分项的极点

实标量场的 Feynman 传播子

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🍰 $D_F(x - y)$ 是 Lorentz 不变的，而且是一个偶函数：

$$D_F(y - x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{ip \cdot (x - y)}}{p^2 - m^2 + i\epsilon} = \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip \cdot (x - y)}}{p^2 - m^2 + i\epsilon} = D_F(x - y)$$

🌀 第二步作了变量替换 $p^\mu \rightarrow -p^\mu$

🍰 可见， $\overline{\phi(y)\phi(x)} = \overline{\phi(x)\phi(y)}$


6.4.2 小节 复标量场的 Feynman 传播子


 在相互作用绘景中，将复标量场 $\phi(x)$ 的平面波展开式分解为正负能解部分，得

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x), \quad \phi^\dagger(x) = \phi^{\dagger(+)}(x) + \phi^{\dagger(-)}(x)$$

$$\phi^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ip \cdot x}, \quad \phi^{\dagger(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} b_p e^{-ip \cdot x}$$

$$\phi^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} b_p^\dagger e^{ip \cdot x}, \quad \phi^{\dagger(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger e^{ip \cdot x}$$

 注意场算符缩并的定义是 $\overline{\Phi_a(x)\Phi_b(y)} = \begin{cases} [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}, & x^0 \geq y^0 \\ \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}, & x^0 < y^0 \end{cases}$

 由于 (a_p, a_p^\dagger) 和 (b_p, b_p^\dagger) 是两套相互独立的产生湮灭算符，有

$$\overline{\phi(x)\phi(y)} = \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle = 0, \quad \overline{\phi^\dagger(x)\phi^\dagger(y)} = \langle 0 | T[\phi^\dagger(x)\phi^\dagger(y)] | 0 \rangle = 0$$

 复标量场的 Feynman 传播子是非平庸的缩并，定义为

$$D_F(x-y) \equiv \overline{\phi(x)\phi^\dagger(y)} = \langle 0 | T[\phi(x)\phi^\dagger(y)] | 0 \rangle$$

$\langle 0 | \mathbf{T}[\phi(x)\phi^\dagger(y)] | 0 \rangle$

类似于实标量场的计算，利用产生湮灭算符的对易关系，得到


$$\begin{aligned} \langle 0 | \phi(x)\phi^\dagger(y) | 0 \rangle &= \langle 0 | \phi^{(+)}(x)\phi^{(-)}(y) | 0 \rangle = \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{4E_p E_q}} \langle 0 | a_p e^{-ip \cdot x} a_q^\dagger e^{iq \cdot y} | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{4E_p E_q}} \langle 0 | [a_p, a_q^\dagger] | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^3 \sqrt{4E_p E_q}} \delta^{(3)}(\mathbf{p} - \mathbf{q}) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip \cdot (x-y)}}{2E_p} \end{aligned}$$

$$\begin{aligned} \langle 0 | \phi^\dagger(y)\phi(x) | 0 \rangle &= \langle 0 | \phi^{\dagger(+)}(y)\phi^{(-)}(x) | 0 \rangle = \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{4E_p E_q}} \langle 0 | b_p e^{-ip \cdot y} b_q^\dagger e^{iq \cdot x} | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot y - q \cdot x)}}{(2\pi)^6 \sqrt{4E_p E_q}} \langle 0 | [b_p, b_q^\dagger] | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot y - q \cdot x)}}{(2\pi)^3 \sqrt{4E_p E_q}} \delta^{(3)}(\mathbf{p} - \mathbf{q}) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip \cdot (x-y)}}{2E_p} \end{aligned}$$


👉 $\langle 0 | \mathbf{T}[\phi(x)\phi^\dagger(y)] | 0 \rangle = \theta(x^0 - y^0) \langle 0 | \phi(x)\phi^\dagger(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi^\dagger(y)\phi(x) | 0 \rangle$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}]$$


四维动量积分


 **归纳**上面实标量场计算中的相关结果，有

$$\theta(x^0 - y^0) \frac{e^{-iE_p(x^0 - y^0)}}{2E_p} + \theta(y^0 - x^0) \frac{e^{iE_p(x^0 - y^0)}}{2E_p} = \int \frac{dp^0}{2\pi} \frac{i e^{-ip^0(x^0 - y^0)}}{p^2 - m^2 + i\epsilon}$$

 **进而推出**

$$\begin{aligned} \langle 0 | T[\phi(x)\phi^\dagger(y)] | 0 \rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ &= \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot (x-y)} \left[\theta(x^0 - y^0) \frac{e^{-iE_p(x^0 - y^0)}}{2E_p} + \theta(y^0 - x^0) \frac{e^{iE_p(x^0 - y^0)}}{2E_p} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot (x-y)} \int \frac{dp^0}{2\pi} \frac{i e^{-ip^0(x^0 - y^0)}}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} \end{aligned}$$

 第二步将方括号第二项积分变量 **p** 替换成 **-p**，从而把 $e^{ip \cdot (x-y)}$ 中的 $e^{-ip \cdot (x-y)}$ 因子变成 $e^{ip \cdot (x-y)}$ **因子**


 最后一步将三维动量积分和 p^0 积分合成**四维动量积分**

复标量场的 Feynman 传播子


 于是，复标量场的 Feynman 传播子表达为

$$\begin{aligned} D_F(x-y) &= \overline{\phi(x)\phi^\dagger(y)} = \langle 0 | T[\phi(x)\phi^\dagger(y)] | 0 \rangle \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)} \end{aligned}$$


 可见，复标量场与实标量场具有相同形式的 Feynman 传播子

 此外，由 $T[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab} T[\Phi_b(y)\Phi_a(x)]$ 和 $D_F(x-y)$ 的偶函数性质得

$$\overline{\phi^\dagger(x)\phi(y)} = \langle 0 | T[\phi^\dagger(x)\phi(y)] | 0 \rangle = \langle 0 | T[\phi(y)\phi^\dagger(x)] | 0 \rangle = D_F(y-x) = D_F(x-y)$$

 也就是说， $\overline{\phi^\dagger(x)\phi(y)}$ 与 $\overline{\phi(x)\phi^\dagger(y)}$ 相等

6.4.3 小节 有质量矢量场的 Feynman 传播子

 有质量实矢量场 $A^\mu(x)$ 的 Feynman 传播子 $\Delta_F^{\mu\nu}(x-y)$ 定义为

$$\Delta_F^{\mu\nu}(x-y) \equiv \overbrace{A^\mu(x)A^\nu(y)} = \langle 0 | T[A^\mu(x)A^\nu(y)] | 0 \rangle$$

 根据平面波展开式、产生湮灭算符的对易关系和极化矢量求和关系，有

$$\begin{aligned} \langle 0 | A^\mu(x)A^\nu(y) | 0 \rangle &= \langle 0 | A^{\mu(+)}(x)A^{\nu(-)}(y) | 0 \rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{4E_p E_q}} \sum_{\lambda\lambda'} \langle 0 | \varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} \varepsilon^{\nu*}(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot y} | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{4E_p E_q}} \sum_{\lambda\lambda'} \varepsilon^\mu(\mathbf{p}, \lambda) \varepsilon^{\nu*}(\mathbf{q}, \lambda') \langle 0 | [a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger] | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^3 \sqrt{4E_p E_q}} \sum_{\lambda\lambda'} \varepsilon^\mu(\mathbf{p}, \lambda) \varepsilon^{\nu*}(\mathbf{q}, \lambda') \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= \int \frac{d^3p e^{-ip \cdot (x-y)}}{(2\pi)^3 2E_p} \sum_{\lambda} \varepsilon^\mu(\mathbf{p}, \lambda) \varepsilon^{\nu*}(\mathbf{p}, \lambda) = \int \frac{d^3p}{(2\pi)^3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right) \frac{e^{-ip \cdot (x-y)}}{2E_p} \end{aligned}$$

$\langle 0 | T[A^\mu(x)A^\nu(y)] | 0 \rangle$

👑 由上述 $\langle 0 | A^\mu(x)A^\nu(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right) \frac{e^{-ip \cdot (x-y)}}{2E_p}$ 得到

$$\begin{aligned} \langle 0 | A^\nu(y)A^\mu(x) | 0 \rangle &= \int \frac{d^3p}{(2\pi)^3} \left(-g^{\nu\mu} + \frac{p^\nu p^\mu}{m^2} \right) \frac{e^{-ip \cdot (y-x)}}{2E_p} \\ &= \int \frac{d^3p}{(2\pi)^3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right) \frac{e^{ip \cdot (x-y)}}{2E_p} \end{aligned}$$

👑 从而推出 Feynman 传播子表达式

$$\begin{aligned} \Delta_F^{\mu\nu}(x-y) &= \langle 0 | T[A^\mu(x)A^\nu(y)] | 0 \rangle \\ &= \theta(x^0 - y^0) \langle 0 | A^\mu(x)A^\nu(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | A^\nu(y)A^\mu(x) | 0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right) \frac{1}{2E_p} [\theta(x^0 - y^0)e^{-ip \cdot (x-y)} + \theta(y^0 - x^0)e^{ip \cdot (x-y)}] \end{aligned}$$

👑 最后一行圆括号中的项 $p^\mu p^\nu / m^2$ 与 p^0 有关，因此不能直接应用前面的

$$\theta(x^0 - y^0) \frac{e^{-iE_p(x^0 - y^0)}}{2E_p} + \theta(y^0 - x^0) \frac{e^{iE_p(x^0 - y^0)}}{2E_p} = \int \frac{dp^0}{2\pi} \frac{i e^{-ip^0(x^0 - y^0)}}{p^2 - m^2 + i\epsilon}$$

合成四维动量积分

求导运算

👉 为了得到简洁的表达式，需要将 $p^\mu p^\nu / m^2$ 转换为求导运算

👤 记 $\partial_x^\mu \equiv \partial / \partial x_\mu$ ，利用阶跃函数与 δ 函数的关系 $\theta'(x) = \delta(x)$ ，推出

$$\begin{aligned}
 & \partial_x^\mu \partial_x^\nu [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\
 = & \partial_x^\mu [-ip^\nu \theta(x^0 - y^0) e^{-ip \cdot (x-y)} + g^{\nu 0} \delta(x^0 - y^0) e^{-ip \cdot (x-y)} + ip^\nu \theta(y^0 - x^0) e^{ip \cdot (x-y)} \\
 & - g^{\nu 0} \delta(y^0 - x^0) e^{ip \cdot (x-y)}] \\
 = & -p^\mu p^\nu \theta(x^0 - y^0) e^{-ip \cdot (x-y)} - ig^{\mu 0} p^\nu \delta(x^0 - y^0) e^{-ip \cdot (x-y)} - ip^\mu g^{\nu 0} \delta(x^0 - y^0) e^{-ip \cdot (x-y)} \\
 & + g^{\mu 0} g^{\nu 0} \partial_x^0 \delta(x^0 - y^0) e^{-ip \cdot (x-y)} - p^\mu p^\nu \theta(y^0 - x^0) e^{ip \cdot (x-y)} - ig^{\mu 0} p^\nu \delta(y^0 - x^0) e^{ip \cdot (x-y)} \\
 & - ip^\mu g^{\nu 0} \delta(y^0 - x^0) e^{ip \cdot (x-y)} + g^{\mu 0} g^{\nu 0} \partial_x^0 \delta(y^0 - x^0) e^{ip \cdot (x-y)} \\
 = & -p^\mu p^\nu [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\
 & - i(g^{\mu 0} p^\nu + g^{\nu 0} p^\mu) \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} + e^{ip \cdot (x-y)}] \\
 & + g^{\mu 0} g^{\nu 0} \partial_x^0 \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}]
 \end{aligned}$$

分解

🎓 从而将包含 $p^\mu p^\nu / m^2$ 的因子化为

$$\begin{aligned} & \frac{p^\mu p^\nu}{m^2} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ = & -\frac{\partial_x^\mu \partial_x^\nu}{m^2} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ & -\frac{i}{m^2} (g^{\mu 0} p^\nu + g^{\nu 0} p^\mu) \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} + e^{ip \cdot (x-y)}] \\ & +\frac{g^{\mu 0} g^{\nu 0}}{m^2} \partial_x^0 \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}] \end{aligned}$$

👤 由此将 Feynman 传播子分解成 $\Delta_F^{\mu\nu}(x-y) = f_1^{\mu\nu}(x,y) + f_2^{\mu\nu}(x,y) + f_3^{\mu\nu}(x,y)$

$$f_1^{\mu\nu}(x,y) \equiv -\left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2}\right) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}]$$

$$f_2^{\mu\nu}(x,y) \equiv -\frac{i}{m^2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (g^{\mu 0} p^\nu + g^{\nu 0} p^\mu) \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} + e^{ip \cdot (x-y)}]$$

$$f_3^{\mu\nu}(x,y) \equiv \frac{g^{\mu 0} g^{\nu 0}}{m^2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \partial_x^0 \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}]$$

$f_1^{\mu\nu}(x, y)$ 和 $f_2^{\mu\nu}(x, y)$

 将**第一项**化为

$$\begin{aligned}
 f_1^{\mu\nu}(x, y) &= - \left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2} \right) \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\
 &= - \left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2} \right) \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} = \int \frac{d^4 p}{(2\pi)^4} \frac{-i(g^{\mu\nu} - p^\mu p^\nu / m^2)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}
 \end{aligned}$$


 $\delta(x^0 - y^0)$ 只在 $x^0 - y^0 = 0$ 处非零，该处有 $e^{-iE_p(x^0 - y^0)} = e^{iE_p(x^0 - y^0)} = 1$ ，故

$$f_2^{i0}(x, y) = f_2^{0i}(x, y) = -\frac{i}{m^2} \int \frac{d^3 p}{(2\pi)^3} \frac{p^i}{2E_p} \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} + e^{ip \cdot (x-y)}] = 0$$

 上式中被积函数是关于 \mathbf{p} 的奇函数，因而对整个三维动量空间积分为零

 利用 **Fourier 变换公式**导出


$$\begin{aligned}
 f_2^{00}(x, y) &= -\frac{i}{m^2} \int \frac{d^3 p}{(2\pi)^3} \frac{2p^0}{2E_p} \delta(x^0 - y^0) [e^{ip \cdot (x-y)} + e^{-ip \cdot (x-y)}] \\
 &= -\frac{2i}{m^2} \delta(x^0 - y^0) \delta^{(3)}(\mathbf{x} - \mathbf{y}) = -\frac{2i}{m^2} \delta^{(4)}(x - y)
 \end{aligned}$$

 归纳得到**第二项**表达式 $f_2^{\mu\nu}(x, y) = -\frac{2i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x - y)$

$f_3^{\mu\nu}(x, y)$

 另一方面，根据 δ 函数的导数的定义，有

$$\int dx f(x)\delta'(x-a) = -f'(a) = -\int dx f'(x)\delta(x-a)$$

 因而对第三项中的被积函数可作替换

$$\partial_x^0 \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}] \rightarrow -\delta(x^0 - y^0) \partial_x^0 [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}]$$

 由此将第三项化为


$$\begin{aligned} f_3^{\mu\nu}(x, y) &= \frac{g^{\mu 0} g^{\nu 0}}{m^2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \partial_x^0 \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}] \\ &= -\frac{g^{\mu 0} g^{\nu 0}}{m^2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \delta(x^0 - y^0) \partial_x^0 [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}] \\ &= -\frac{g^{\mu 0} g^{\nu 0}}{m^2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \delta(x^0 - y^0) [-ip^0 e^{-ip \cdot (x-y)} - ip^0 e^{ip \cdot (x-y)}] \\ &= \frac{i}{2m^2} g^{\mu 0} g^{\nu 0} \int \frac{d^3 p}{(2\pi)^3} \delta(x^0 - y^0) [e^{ip \cdot (x-y)} + e^{-ip \cdot (x-y)}] \\ &= \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta(x^0 - y^0) \delta^{(3)}(\mathbf{x} - \mathbf{y}) = \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x - y) \end{aligned}$$


有质量矢量场的 Feynman 传播子

 综合起来，有质量矢量场 Feynman 传播子的表达式为

$$\begin{aligned}\Delta_{\text{F}}^{\mu\nu}(x-y) &= f_1^{\mu\nu}(x,y) + f_2^{\mu\nu}(x,y) + f_3^{\mu\nu}(x,y) \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{-i(g^{\mu\nu} - p^\mu p^\nu / m^2)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)} - \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x-y)\end{aligned}$$

 第一项是 Lorentz 张量，因而是 Lorentz 协变的，但第二项是非协变的

 幸好，7.4 节将证明这个非协变项在微扰论中的贡献被相互作用哈密顿量密度中非协变项 $\mathcal{H}_{J^0}^1 = \frac{g^2}{2m^2} (J^{1,0})^2$ 的贡献精确抵消，从而理论是 Lorentz 协变的


 因此，在实际计算中可以只保留协变项

$$\Delta_{\text{F}}^{\mu\nu}(x-y) = \langle 0 | \text{T}[A^\mu(x)A^\nu(y)] | 0 \rangle \rightarrow \int \frac{d^4p}{(2\pi)^4} \frac{-i(g^{\mu\nu} - p^\mu p^\nu / m^2)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

6.4.4 小节 无质量矢量场的 Feynman 传播子

 无质量实矢量场 $A^\mu(x)$ 的 Feynman 传播子依赖于规范的选择

 这里取 Feynman 规范 ($\xi = 1$)

 在相互作用绘景中，将无质量实矢量场 $A^\mu(x)$ 的平面波展开式分解为

$$\text{正能解 } A^{\mu(+)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\sigma=0}^3 e^{\mu}(\mathbf{p}, \sigma) b_{\mathbf{p}, \sigma} e^{-ip \cdot x}$$

$$\text{负能解 } A^{\mu(-)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\sigma=0}^3 e^{\mu}(\mathbf{p}, \sigma) b_{\mathbf{p}, \sigma}^\dagger e^{ip \cdot x}$$

 相应的 Feynman 传播子定义为

$$\Delta_F^{\mu\nu}(x-y) \equiv \overbrace{A^\mu(x)A^\nu(y)} = \langle 0 | T[A^\mu(x)A^\nu(y)] | 0 \rangle$$

$\langle 0 | A^\mu(x) A^\nu(y) | 0 \rangle$

根据产生湮灭算符的对易关系和极化矢量的完备性关系，得到

$$\begin{aligned}
 \langle 0 | A^\mu(x) A^\nu(y) | 0 \rangle &= \langle 0 | A^{\mu(+)}(x) A^{\nu(-)}(y) | 0 \rangle \\
 &= \int \frac{d^3 p d^3 q}{(2\pi)^6 \sqrt{4E_p E_q}} \sum_{\sigma \sigma'} \langle 0 | e^{\mu}(\mathbf{p}, \sigma) b_{\mathbf{p}, \sigma} e^{-i\mathbf{p} \cdot x} e^{\nu}(\mathbf{q}, \sigma') b_{\mathbf{q}, \sigma'}^\dagger e^{i\mathbf{q} \cdot y} | 0 \rangle \\
 &= \int \frac{d^3 p d^3 q e^{-i(\mathbf{p} \cdot x - \mathbf{q} \cdot y)}}{(2\pi)^6 \sqrt{4E_p E_q}} \sum_{\sigma \sigma'} e^{\mu}(\mathbf{p}, \sigma) e^{\nu}(\mathbf{q}, \sigma') \langle 0 | [b_{\mathbf{p}, \sigma}, b_{\mathbf{q}, \sigma'}^\dagger] | 0 \rangle \\
 &= - \int \frac{d^3 p d^3 q e^{-i(\mathbf{p} \cdot x - \mathbf{q} \cdot y)}}{(2\pi)^3 \sqrt{4E_p E_q}} \sum_{\sigma \sigma'} e^{\mu}(\mathbf{p}, \sigma) e^{\nu}(\mathbf{q}, \sigma') g_{\sigma \sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\
 &= - \int \frac{d^3 p e^{-i\mathbf{p} \cdot (x-y)}}{(2\pi)^3 2E_p} \sum_{\sigma} g_{\sigma \sigma} e^{\mu}(\mathbf{p}, \sigma) e^{\nu}(\mathbf{p}, \sigma) = -g^{\mu\nu} \int \frac{d^3 p}{(2\pi)^3} \frac{e^{-i\mathbf{p} \cdot (x-y)}}{2E_p}
 \end{aligned}$$

进而推出

$$\langle 0 | A^\nu(y) A^\mu(x) | 0 \rangle = -g^{\nu\mu} \int \frac{d^3 p}{(2\pi)^3} \frac{e^{-i\mathbf{p} \cdot (y-x)}}{2E_p} = -g^{\mu\nu} \int \frac{d^3 p}{(2\pi)^3} \frac{e^{i\mathbf{p} \cdot (x-y)}}{2E_p}$$

无质量矢量场的 Feynman 传播子

👉 从而得到

$$\begin{aligned} \Delta_F^{\mu\nu}(x-y) &= \langle 0 | T[A^\mu(x)A^\nu(y)] | 0 \rangle \\ &= \theta(x^0 - y^0) \langle 0 | A^\mu(x)A^\nu(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | A^\nu(y)A^\mu(x) | 0 \rangle \\ &= -g^{\mu\nu} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [\theta(x^0 - y^0)e^{-ip \cdot (x-y)} + \theta(y^0 - x^0)e^{ip \cdot (x-y)}] \end{aligned}$$


👉 当质量 $m = 0$ 时, 有

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [\theta(x^0 - y^0)e^{-ip \cdot (x-y)} + \theta(y^0 - x^0)e^{ip \cdot (x-y)}] = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{p^2 + i\epsilon}$$

👉 于是, **Feynman 规范**下无质量矢量场的 **Feynman 传播子**可以表达为

$$\Delta_F^{\mu\nu}(x-y) = \langle 0 | T[A^\mu(x)A^\nu(y)] | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-ig^{\mu\nu}}{p^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

6.4.5 小节 Dirac 旋量场的 Feynman 传播子

 在相互作用绘景中，将 $\psi_a(x)$ 和 $\bar{\psi}_a(x)$ 的平面波展开式分解为**正负**能解部分，得

$$\psi_a(x) = \psi_a^{(+)}(x) + \psi_a^{(-)}(x), \quad \bar{\psi}_a(x) = \bar{\psi}_a^{(+)}(x) + \bar{\psi}_a^{(-)}(x)$$

$$\psi_a^{(+)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm} u_a(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x}$$

$$\psi_a^{(-)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm} v_a(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x}$$

$$\bar{\psi}_a^{(+)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm} \bar{v}_a(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda} e^{-ip \cdot x}$$

$$\bar{\psi}_a^{(-)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm} \bar{u}_a(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x}$$

 容易看出， $\overbrace{\psi_a(x)\psi_b(y)} = \overbrace{\bar{\psi}_a(x)\bar{\psi}_b(y)} = 0$

 Dirac 旋量场 $\psi_a(x)$ 的 Feynman 传播子 $S_{F,ab}(x-y)$ 定义为


$$S_{F,ab}(x-y) \equiv \overbrace{\psi_a(x)\bar{\psi}_b(y)} = \langle 0 | T[\psi_a(x)\bar{\psi}_b(y)] | 0 \rangle$$

$$\langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle$$

 利用产生湮灭算符的反对易关系和自旋求和关系，推出

$$\begin{aligned} & \langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle \\ &= \langle 0 | \psi_a^{(+)}(x) \bar{\psi}_b^{(-)}(y) | 0 \rangle \\ &= \int \frac{d^3 p d^3 q}{(2\pi)^6 \sqrt{4E_p E_q}} \sum_{\lambda \lambda'} \langle 0 | u_a(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} \bar{u}_b(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot y} | 0 \rangle \\ &= \int \frac{d^3 p d^3 q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{4E_p E_q}} \sum_{\lambda \lambda'} u_a(\mathbf{p}, \lambda) \bar{u}_b(\mathbf{q}, \lambda') \langle 0 | \{a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger\} | 0 \rangle \\ &= \int \frac{d^3 p d^3 q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^3 \sqrt{4E_p E_q}} \sum_{\lambda \lambda'} u_a(\mathbf{p}, \lambda) \bar{u}_b(\mathbf{q}, \lambda') \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= \int \frac{d^3 p e^{-ip \cdot (x-y)}}{(2\pi)^3 2E_p} \sum_{\lambda} u_a(\mathbf{p}, \lambda) \bar{u}_b(\mathbf{p}, \lambda) \\ &= \int \frac{d^3 p}{(2\pi)^3} (\not{p} + m)_{ab} \frac{e^{-ip \cdot (x-y)}}{2E_p} \end{aligned}$$

$$\langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle$$

 再推出

$$\begin{aligned} & \langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle \\ &= \langle 0 | \bar{\psi}_b^{(+)}(y) \psi_a^{(-)}(x) | 0 \rangle \\ &= \int \frac{d^3 p d^3 q}{(2\pi)^6 \sqrt{4E_p E_q}} \sum_{\lambda \lambda'} \langle 0 | \bar{v}_b(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-i p \cdot y} v_a(\mathbf{q}, \lambda') b_{\mathbf{q}, \lambda'}^\dagger e^{i q \cdot x} | 0 \rangle \\ &= \int \frac{d^3 p d^3 q e^{-i(p \cdot y - q \cdot x)}}{(2\pi)^6 \sqrt{4E_p E_q}} \sum_{\lambda \lambda'} v_a(\mathbf{q}, \lambda') \bar{v}_b(\mathbf{p}, \lambda) \langle 0 | \{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} | 0 \rangle \\ &= \int \frac{d^3 p d^3 q e^{-i(p \cdot y - q \cdot x)}}{(2\pi)^3 \sqrt{4E_p E_q}} \sum_{\lambda \lambda'} v_a(\mathbf{q}, \lambda') \bar{v}_b(\mathbf{p}, \lambda) \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= \int \frac{d^3 p e^{-i p \cdot (y - x)}}{(2\pi)^3 2E_p} \sum_{\lambda} v_a(\mathbf{p}, \lambda) \bar{v}_b(\mathbf{p}, \lambda) \\ &= \int \frac{d^3 p}{(2\pi)^3} (\not{p} - m)_{ab} \frac{e^{i p \cdot (x - y)}}{2E_p} \end{aligned}$$

Dirac 旋量场的 Feynman 传播子

✂ 于是, Dirac 旋量场的 Feynman 传播子为

$$\begin{aligned}
 S_{F,ab}(x-y) &= \langle 0 | T[\psi_a(x)\bar{\psi}_b(y)] | 0 \rangle \\
 &= \theta(x^0 - y^0) \langle 0 | \psi_a(x)\bar{\psi}_b(y) | 0 \rangle - \theta(y^0 - x^0) \langle 0 | \bar{\psi}_b(y)\psi_a(x) | 0 \rangle \\
 &= \int \frac{d^3p}{(2\pi)^3} \left[\theta(x^0 - y^0) (\gamma_\mu p^\mu + m)_{ab} \frac{e^{-ip \cdot (x-y)}}{2E_p} \right. \\
 &\quad \left. - \theta(y^0 - x^0) (\gamma_\mu p^\mu - m)_{ab} \frac{e^{ip \cdot (x-y)}}{2E_p} \right] \\
 &= (\gamma_0)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [E_p \theta(x^0 - y^0) e^{-ip \cdot (x-y)} - E_p \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\
 &\quad + (\gamma_i)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [p^i \theta(x^0 - y^0) e^{-ip \cdot (x-y)} - p^i \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\
 &\quad + m \delta_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}]
 \end{aligned}$$

Dirac 旋量场的 Feynman 传播子

🔑 类似于前面得到的等式

$$\theta(x^0 - y^0) \frac{e^{-iE_p(x^0 - y^0)}}{2E_p} + \theta(y^0 - x^0) \frac{e^{iE_p(x^0 - y^0)}}{2E_p} = \int \frac{dp^0}{2\pi} \frac{i e^{-ip^0(x^0 - y^0)}}{p^2 - m^2 + i\epsilon}$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}$$

🔑 对于任意解析函数 $f(p^0)$ ，可以推出

$$\theta(x^0 - y^0) \frac{f(E_p) e^{-iE_p(x^0 - y^0)}}{2E_p} + \theta(y^0 - x^0) \frac{f(-E_p) e^{iE_p(x^0 - y^0)}}{2E_p}$$

$$= \int \frac{dp^0}{2\pi} \frac{i f(p^0) e^{-ip^0(x^0 - y^0)}}{p^2 - m^2 + i\epsilon}$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [f(E_p) \theta(x^0 - y^0) e^{-ip \cdot (x-y)} + f(-E_p) \theta(y^0 - x^0) e^{ip \cdot (x-y)}]$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i f(p^0) e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}$$

🔑 左边**第一项**对应于 $p^0 = E_p$ 处的**留数**，**第二项**对应于 $p^0 = -E_p$ 处的**留数**

Dirac 旋量场的 Feynman 传播子

🔓 取 $f(p^0) = p^0$, 把 $S_{F,ab}(x-y)$ 的第一项变成

$$\begin{aligned} (\gamma_0)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [E_p \theta(x^0 - y^0) e^{-ip \cdot (x-y)} - E_p \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ = \int \frac{d^4p}{(2\pi)^4} \frac{i(\gamma_0 p^0)_{ab} e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} \end{aligned}$$

🔓 再将 $S_{F,ab}(x-y)$ 式的后两项化成

$$\begin{aligned} (\gamma_i)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [p^i \theta(x^0 - y^0) e^{-ip \cdot (x-y)} - p^i \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ + m\delta_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ = [(\gamma_i)_{ab} i\partial_x^i + m\delta_{ab}] \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ = [(\gamma_i)_{ab} i\partial_x^i + m\delta_{ab}] \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\gamma_i p^i + m)_{ab}}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}. \end{aligned}$$

🖋️ 把这三项加起来, 得到 Dirac 旋量场的 Feynman 传播子

$$S_{F,ab}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)_{ab}}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

等价形式

将 Dirac 旋量场的 Feynman 传播子写成旋量空间矩阵的形式，得

$$S_F(x-y) = \langle 0 | T[\psi(x)\bar{\psi}(y)] | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

由 $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ 有

$$\not{p}\not{p} = p_\mu p_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} p_\mu p_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = p_\mu p_\nu g^{\mu\nu} = p^2$$

从而得到 $(\not{p} + m)(\not{p} - m) = \not{p}\not{p} - m^2 = p^2 - m^2$ ，则

$$(\not{p} + m)(\not{p} - m + i\epsilon) = p^2 - m^2 + i\epsilon(\not{p} + m)$$

$i\epsilon(\not{p} + m)$ 是无穷小量，因而上式右边与 $p^2 - m^2 + i\epsilon$ 等价，故

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m) e^{-ip \cdot (x-y)}}{(\not{p} + m)(\not{p} - m + i\epsilon)} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p} - m + i\epsilon} e^{-ip \cdot (x-y)}$$

最后的表达式更为简洁，但在矩阵的意义上不好理解，应将它转化回原式来理解