Gamma-ray	Collider sensitivity	<b>Beam polarization</b>	Unitarity bounds	Conclusions	Backups 00000000

## Potential dark matter $\gamma$ -ray line signature observed by Fermi-LAT and its test at high energy $e^+e^-$ colliders

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## $\gamma$ -ray emission from DM: continuous spectrum

**Dark matter** (DM,  $\chi$ ) pair annihilation or decay into  $e^+e^-, \mu^+\mu^-, \tau^+\tau^-, q\bar{q}, W^+W^-, Z^0Z^0, h^0h^0$  **Gamma-ray emission from final state radiation or decay** 

Cut-off energy:  $m_{\gamma}$  for DM annihilation,  $m_{\gamma}/2$  for DM decay



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**Dark matter** (DM,  $\chi$ ) pair annihilation or decay into  $e^+e^-, \mu^+\mu^-, \tau^+\tau^-, q\bar{q}, W^+W^-, Z^0Z^0, h^0h^0$   $\downarrow \downarrow$ **Gamma-ray emission from final state radiation or decay** 

Cut-off energy:  $m_{\gamma}$  for DM annihilation,  $m_{\gamma}/2$  for DM decay

Searching for DM signature in DM-dominant regions:

Galactic center Galactic halo dwarf spheroidal galaxies clusters of galaxies



Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups
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## $\gamma$ -ray emission from DM: line spectrum

In general, DM particles  $(\chi)$  should not have electric charge and not directly couple to photons  $\downarrow\downarrow$ **DM particles may couple to photons via high order loop diagrams** (highly suppressed, the branching fraction may be only ~  $10^{-4} - 10^{-1}$ )



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## A $\gamma$ -ray line from the Galactic center region?

Using the 3.7-year Fermi-LAT  $\gamma$ -ray data, several analyses showed that there might be evidence of **a monochromatic**  $\gamma$ -ray line at energy  $\sim 130$  GeV, originating from the Galactic center region (about  $3 - 4\sigma$ ). It may be due to DM annihilation with  $\langle \sigma_{ann} v \rangle \sim 10^{-27} \text{ cm}^3 \text{ s}^{-1}$ .



Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups
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Recently, the Fermi-LAT Collaboration has released its official spectral line search in the energy range 5 - 300 GeV using 3.7 years of data.

They **did not find any globally significant lines** and set 95% CL upper limits for DM annihilation cross sections.

Their most significant fit occurred at  $E_{\gamma} = 133$  GeV and had **a local** significance of 3.3 $\sigma$ , which translates to a global significance of 1.6 $\sigma$ .



Fermi-LAT Collaboration, 1305.5597

Gamma-ray	Collider sensitivity •00000	Beam polarization	Unitarity bounds	Conclusions	Backups 00000000

## DM-photon interaction at $e^+e^-$ colliders



The coupling between DM particles and photons that induce the annihilation process  $\chi \chi \rightarrow \gamma \gamma$  can also lead to the process  $e^+e^- \rightarrow \chi \chi \gamma$ . Therefore, the possible  $\gamma$ -ray line signal observed by Fermi-LAT may be tested at future TeV-scale  $e^+e^-$  colliders.

#### DM particles escape from the detector

**Signature:** a monophoton associating with missing energy  $(\gamma + \not E)$ 

Gamma-ray	Collider sensitivity ○●○○○○	Beam polarization	Unitarity bounds	Conclusions	Backups

## Effective operator approach

If DM particles couple to photons via exchanging some mediators which are **sufficiently heavy**, the DM-photon coupling can be approximately described by **effective contact operators**.

For Dirac fermionic DM, consider  $\mathcal{O}_F = \frac{1}{\sqrt{3}} \bar{\chi} i \gamma_5 \chi F_{\mu\nu} \tilde{F}^{\mu\nu}$ :  $\langle \sigma_{\rm ann} v \rangle_{\chi \bar{\chi} \to 2\gamma} \simeq \frac{4m_{\chi}^4}{\pi \Lambda^6}, \qquad \sigma(e^+ e^- \to \chi \bar{\chi} \gamma) \sim \frac{s^2}{\Lambda^6}$ Fermi  $\gamma$ -ray line signal  $\iff m_{\gamma} \simeq 130$  GeV,  $\Lambda \sim 1$  TeV For complex scalar DM, consider  $O_S = \frac{1}{\Lambda^2} \chi^* \chi F_{\mu\nu} F^{\mu\nu}$ :  $\langle \sigma_{\rm ann} v \rangle_{\chi \chi^* \to 2\gamma} \simeq \frac{2m_{\chi}^2}{\pi \Lambda^4}, \quad \sigma(e^+ e^- \to \chi \chi^* \gamma) \sim \frac{s}{\Lambda^4}$ Fermi  $\gamma$ -ray line signal  $\iff m_{\gamma} \simeq 130$  GeV,  $\Lambda \sim 3$  TeV

Gamma-ray 0000	Collider sensitivity ○○●○○○	Beam polarization	Unitarity bounds	Conclusions	Backups 00000000

In the  $\gamma + \not\!\!\!E$  searching channel, the main background is  $e^+e^- \rightarrow v \bar{v} \gamma$ :



Minor backgrounds:  $e^+e^- \rightarrow e^+e^-\gamma$ ,  $e^+e^- \rightarrow \tau^+\tau^-\gamma$ , ...

#### **Simulation:** FeynRules $\rightarrow$ MadGraph 5 $\rightarrow$ PGS 4

ILD-like ECAL energy resolution: 
$$\frac{\Delta E}{E} = \frac{16.6\%}{\sqrt{E/\text{GeV}}} \oplus 1.1\%$$

Future  $e^+e^-$  colliders:  $\sqrt{s} = 250 \text{ GeV}$  ("Higgs factory"),  $\sqrt{s} = 500 \text{ GeV}$  (typical ILC),  $\sqrt{s} = 1 \text{ TeV}$  (upgraded ILC & initial CLIC),  $\sqrt{s} = 3 \text{ TeV}$  (ultimate CLIC)

Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups
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**Cut 1 (pre-selection):** Require a photon with  $E_{\gamma} > 10 \text{ GeV}$ and  $10^{\circ} < \theta_{\gamma} < 170^{\circ}$ Veto any other particle

**Benchmark point:**  $\Lambda = 200 \text{ GeV}, m_{\gamma} = 100(50) \text{ GeV}$  for fermionic (scalar) DM

Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups
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**Cut 1 (pre-selection):** Require a photon with  $E_{\gamma} > 10 \text{ GeV}$ and  $10^{\circ} < \theta_{\gamma} < 170^{\circ}$ Veto any other particle

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Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups
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Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups
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**Cut 1 (pre-selection):** Require a photon with  $E_{\gamma} > 10$  GeV and  $10^{\circ} < \theta_{\gamma} < 170^{\circ}$ Veto any other particle **Cut 2:** Veto 50 GeV  $< m_{\text{miss}} < 130$  GeV **Cut 3:** Require  $30^{\circ} < \theta_{\gamma} < 150^{\circ}$ **Cut 4:** Require  $p_{\tau}^{\gamma} > \sqrt{s}/10$ 

**Benchmark point:**  $\Lambda = 200$  GeV,  $m_{\gamma} = 100(50)$  GeV for fermionic (scalar) DM

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#### Cross sections and signal significances after each cut

	$v \bar{v} \gamma$	$e^+e^-\gamma$	Fermior	nic DM	Scala	r DM
	$\sigma$ (fb)	$\sigma$ (fb)	$\sigma$ (fb)	$S/\sqrt{B}$	$\sigma$ (fb)	$S/\sqrt{B}$
Cut 1	2415.2	173.0	646.8	12.7	321.4	6.3
Cut 2	2102.5	168.6	646.8	13.6	308.2	6.5
Cut 3	1161.1	16.8	538.0	15.7	255.9	7.5
Cut 4	254.5	1.9	520.7	32.5	253.9	15.8

Benchmark point:  $\Lambda = 200$  GeV,  $m_{\gamma} = 100(50)$  GeV for fermionic (scalar) DM

#### Most of the signal events remain

 $e^+e^- \rightarrow v\bar{v}\gamma$  background: reduced by almost **an order of magnitude**  $e^+e^- \rightarrow e^+e^-\gamma$  background: only **one percent** survives

$$(\sqrt{s} = 500 \text{ GeV}, 1 \text{ fb}^{-1})$$



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Gamma-ray	Collider sensitivity	Beam polarization ●○	Unitarity bounds	Conclusions	Backups
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### Beam polarization

For a process at an  $e^+e^-$  collider with **polarized beams**,

$$\sigma(P_{e^{-}}, P_{e^{+}}) = \frac{1}{4} \left[ (1 + P_{e^{-}})(1 + P_{e^{+}})\sigma_{\mathrm{RR}} + (1 - P_{e^{-}})(1 - P_{e^{+}})\sigma_{\mathrm{LL}} + (1 + P_{e^{-}})(1 - P_{e^{+}})\sigma_{\mathrm{RL}} + (1 - P_{e^{-}})(1 + P_{e^{+}})\sigma_{\mathrm{LR}} \right]$$



▲  $(P_{e^-}, P_{e^+}) = (0.8, -0.3)$  can be achieved at the ILC [ILC technical design report, Vol. 1, 1306.6327]





Using the **polarized beams** is roughly equivalent to **increasing** the integrated luminosity by **an order of magnitude**.

For fermionic DM (scalar DM), a data set of 2000 fb<sup>-1</sup> (1000 fb<sup>-1</sup>) would be just sufficient to test the Fermi  $\gamma$ -ray line signal at an  $e^+e^-$  collider with  $\sqrt{s} = 1$  TeV (3 TeV).

Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds ●○○○○	Conclusions	Backups
S-matri	x unitarity				

For quantum scattering theories,

*S*-matrix unitarity  $(S^{\dagger}S = 1) \iff$  conservation of probability

A process violate the unitarity in a non-renormalizable effective theory ↓ The theory is **invalid** for this process ↓ A **UV-complete theory** may be needed for a full description

The effective operator treatment for DM searches at colliders should be carefully checked by verifying the *S*-matrix unitarity.

Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds ○●○○○	Conclusions	Backups 00000000
Unitarit	v conditio	ns			

The  $2 \rightarrow 2$  amplitude  $\mathcal{M}(\cos \theta)$  can be expanded as **partial waves**:

 $\mathcal{M}(\cos\theta) = 16\pi \sum_{j} (2j+1)a_{j}P_{j}(\cos\theta), \quad a_{j} = \frac{1}{32\pi} \int_{-1}^{1} d\cos\theta P_{j}(\cos\theta)\mathcal{M}(\cos\theta)$ 

Unitarity condition for  $2 \rightarrow 2$  elastic scattering:

$$\left|\operatorname{Re} a_{j}^{\mathrm{el}}\right| \leq \frac{1}{2}, \ \forall j$$

Unitarity condition for  $2 \rightarrow 2$  inelastic scattering:

$$\left|a_{j}^{\mathrm{inel}}\right| \leq \frac{1}{2\sqrt{\beta_{f}}}, \ \forall j$$

 $(\beta_f$  is the velocity of either of the final particles)

Gamma-rav Collider sensitivity Beam polarization Unitarity bounds Conclusions Backups 00000  $S^{\dagger}S = 1$ ,  $S = 1 + iT \implies -i(T - T^{\dagger}) = T^{\dagger}T$  $-i(\mathcal{M}_{\alpha\to\beta}-\mathcal{M}^*_{\beta\to\alpha})=\sum_{\gamma}\int d\Pi_{\gamma}\mathcal{M}^*_{\beta\to\gamma}\mathcal{M}_{\alpha\to\gamma}(2\pi)^4\delta^{(4)}(p_{\alpha}-p_{\gamma})$  $2 \operatorname{Im} \mathcal{M}_{el}(\cos \theta_{\alpha\beta}) = \int d\Pi_{\gamma_{el}} \mathcal{M}^*_{\beta \to \gamma_{el}} \mathcal{M}_{\alpha \to \gamma_{el}}(2\pi)^4 \delta^{(4)}(p_\alpha - p_{\gamma_{el}})$ +  $\int d\Pi_{\gamma_n} \mathcal{M}^*_{\mathcal{B}\to\gamma} \mathcal{M}_{\alpha\to\gamma_n}(2\pi)^4 \delta^{(4)}(p_\alpha - p_{\gamma_n}) + \text{ other inelastic terms}$  $\geq \frac{1}{2\alpha^{-2}} \int d\Omega_{k_1} \mathcal{M}^*_{el}(\cos\theta_{\beta\gamma}) \mathcal{M}_{el}(\cos\theta_{\alpha\gamma}) + \int d\Pi_{\gamma_n} \mathcal{M}^*_{\beta \to \gamma_n} \mathcal{M}_{\alpha \to \gamma_n}(2\pi)^4 \delta^{(4)}(p_\alpha - p_{\gamma_n})$  $\operatorname{Im} a_{i}^{\mathrm{el}} \geq |a_{i}^{\mathrm{el}}|^{2} + |b_{i}^{\mathrm{inel}}|^{2}$  $|b_i^{\text{inel}}|^2 \equiv \frac{1}{64\pi} \int d\cos\theta_{\alpha\beta} P_i(\cos\theta_{\alpha\beta}) \int d\Pi_{\gamma_a} \mathcal{M}^*_{\beta \to \gamma_a} \mathcal{M}_{\alpha \to \gamma_a}(2\pi)^4 \delta^{(4)}(p_\alpha - p_{\gamma_a})$ JΓ

Unitarity condition for any  $2 \rightarrow n$  inelastic scattering:

$$\left|b_{j}^{\mathrm{inel}}\right| \leq \frac{1}{2}, \quad \forall j$$

Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds ○○○●○	Conclusions	Backups

## Unitarity bounds: $2 \rightarrow 2$ vs $2 \rightarrow 3$



Given the same  $\sqrt{s}$ , unitarity bounds for  $2 \rightarrow 2$  scattering are **much** more stringent than those for  $2 \rightarrow 3$  scattering.

However, here the relevant bounds are those for  $2 \rightarrow 3$  scattering.

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All the experimental reaches we obtained lie far beyond the unitarity violation regions.





All the experimental reaches we obtained lie far beyond the unitarity violation regions.

# From the viewpoint of *S*-matrix unitarity, our effective operator treatment do not exceed its valid range.

Gamma-ray	Collider sensitivity	<b>Beam polarization</b>	Unitarity bounds	Conclusions ●○	Backups
Conclus	sions and d	iscussions			

There is a potential γ-ray line signature in the Fermi-LAT data. It can be interpreted as the result of the DM annihilation into photons.
 With a 100 fb<sup>-1</sup> dataset, the potential γ-ray line signature for the fermionic DM can be tested at a 3 TeV e<sup>+</sup>e<sup>-</sup> collider, though the

scalar DM searching would be challenging.

- Using the polarized beams is roughly equivalent to collecting 10 times of data.
- In order to check the validity of the effective operator approach, we derive a general unitarity condition for 2 → n processes. The experimental reaches we obtained are valid since they lie far beyond the unitarity violation regions.

Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions ○●	Backups

# Thanks for your attentions!

Gamma-ray	Collider sensitivity	<b>Beam polarization</b>	Unitarity bounds	Conclusions	Backups ●0000000

# Backup slides

Gamma-ray 0000	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups o●oooooo
Recent	LHC sensi	tivity			

Reinterpreting an ATLAS  $\gamma + \not\!\!\!E_T$  analysis with an integrated luminosity of 4.6 fb<sup>-1</sup> at  $\sqrt{s} = 7$  TeV, Nelson *et al.* [1307.5064] derived constraints on the DM-photon interaction, only up to  $\Lambda \sim 100 - 300$  GeV.



Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups ○○●○○○○○

## An approximate unitarity condition

Note that our unitarity condition  $|b_j^{\text{inel}}| \leq \frac{1}{2}$  is derived without any approximation.

**Through an approximate method**, a unitarity bound on the  $2 \rightarrow n$  inelastic cross section  $\sigma_{\text{inel}}(2 \rightarrow n)$  can be derived to be

$$\sigma_{
m inel}(2
ightarrow n)\leq rac{4\pi}{s}.$$
 [Dicus & H. -J. He, hep-ph/0409131]

We have compared the results given by these two formulas and find that **their differences are rather small** for the processes considered here.

Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups
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#### Unitarity condition in terms of amplitudes:

$$-i(\mathcal{M}_{\alpha\to\beta}-\mathcal{M}^*_{\beta\to\alpha})=\sum_{\gamma}\int d\Pi_{\gamma}\mathcal{M}^*_{\beta\to\gamma}\mathcal{M}_{\alpha\to\gamma}(2\pi)^4\delta^{(4)}(p_{\alpha}-p_{\gamma})$$

For the elastic process  $1 + 2 \rightarrow 1 + 2$ , consider the transitions of state:



Gamma-ray 0000	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups 0000●000

Since  $\mathcal{M}_{\alpha \to \beta} = \mathcal{M}^*_{\beta \to \alpha} = \mathcal{M}_{el}(\cos \theta_{\alpha \beta})$ , the unitarity condition becomes

$$2 \operatorname{Im} \mathcal{M}_{\mathrm{el}}(\cos \theta_{\alpha\beta})$$

$$= \int d\Pi_{\gamma_{\mathrm{el}}} \mathcal{M}^{*}_{\beta \to \gamma_{\mathrm{el}}} \mathcal{M}_{\alpha \to \gamma_{\mathrm{el}}}(2\pi)^{4} \delta^{(4)}(p_{\alpha} - p_{\gamma_{\mathrm{el}}}) + \text{ inelastic terms}$$

$$\geq \frac{\beta_{1}}{32\pi^{2}} \int d\Omega_{k_{1}} \mathcal{M}^{*}_{\mathrm{el}}(\cos \theta_{\beta\gamma}) \mathcal{M}_{\mathrm{el}}(\cos \theta_{\alpha\gamma}),$$

where 
$$\beta_1 \equiv \sqrt{1 - 4m_1^2/s}$$
 and  $d\Omega_{k_1} = d\phi_{k_1} d\cos\theta_{a\gamma}$ .

In terms of partial waves:

$$\operatorname{Im} a_{j}^{\mathrm{el}} \geq \frac{\beta_{1}}{8\pi} \sum_{k,l} (2k+1)(2l+1)a_{k}^{\mathrm{el}*}a_{l}^{\mathrm{el}} \int d\cos\theta_{\alpha\beta} d\Omega_{k_{1}}$$
$$\times P_{j}(\cos\theta_{\alpha\beta})P_{k}(\cos\theta_{\beta\gamma})P_{l}(\cos\theta_{\alpha\gamma})$$

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The addition theorem for Legendre polynomials:

$$P_{k}(\cos \theta_{\beta\gamma}) = P_{k}(\cos \theta_{\alpha\beta})P_{k}(\cos \theta_{\alpha\gamma})$$
  
+2 $\sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!}P_{k}^{m}(\cos \theta_{\alpha\beta})P_{k}^{m}(\cos \theta_{\alpha\gamma})\cos m\phi_{k_{1}}$ 

Carrying out all the integrations, we have

$$\operatorname{Im} a_j^{\text{el}} \ge \beta_1 |a_j^{\text{el}}|^2,$$

which is equivalent to

$$(\operatorname{Re} a_j^{\operatorname{el}})^2 + \left(\operatorname{Im} a_j^{\operatorname{el}} - \frac{1}{2\beta_1}\right)^2 \le \frac{1}{(2\beta_1)^2}.$$

For the scattering of massless particles,  $\beta_1 = 1$ , and it implies

$$\left|\operatorname{Re} a_j^{\operatorname{el}}\right| \leq \frac{1}{2}, \quad \forall j.$$

 $\vec{q_1}$   $\vec{\theta_{\alpha\gamma}}$   $\vec{k_1}$   $\vec{k_1}$   $\vec{q_1}$   $\vec{\theta_{\alpha\beta}}$   $\vec{\theta_{\alpha\gamma}}$   $\vec{k_1}$   $\vec{k_2}$  y

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$$\beta(q_1, q_2) \rightarrow \gamma_n(k_3, \cdots, k_{n+2})$$



Gamma-ray	Collider sensitivity	Beam polarization	Unitarity bounds	Conclusions	Backups 0000000●

The unitarity condition becomes

$$2 \operatorname{Im} \mathcal{M}_{el}(\cos \theta_{\alpha\beta}) = \int d\Pi_{\gamma_{el}} \mathcal{M}^*_{\beta \to \gamma_{el}} \mathcal{M}_{\alpha \to \gamma_{el}} (2\pi)^4 \delta^{(4)}(p_\alpha - p_{\gamma_{el}}) + \int d\Pi_{\gamma_n} \mathcal{M}^*_{\beta \to \gamma_n} \mathcal{M}_{\alpha \to \gamma_n} (2\pi)^4 \delta^{(4)}(p_\alpha - p_{\gamma_n}) + \text{ other inelastic terms} \\ \geq \frac{\beta_1}{32\pi^2} \int d\Omega_{k_1} \mathcal{M}^*_{el} (\cos \theta_{\beta\gamma}) \mathcal{M}_{el} (\cos \theta_{\alpha\gamma}) + \int d\Pi_{\gamma_n} \mathcal{M}^*_{\beta \to \gamma_n} \mathcal{M}_{\alpha \to \gamma_n} (2\pi)^4 \delta^{(4)}(p_\alpha - p_{\gamma_n}).$$

Introducing a new quantity

$$\begin{split} |b_{j}^{\text{inel}}|^{2} &\equiv \frac{1}{64\pi} \int d\cos\theta_{\alpha\beta} P_{j}(\cos\theta_{\alpha\beta}) \int d\Pi_{\gamma_{n}} \mathcal{M}_{\beta \to \gamma_{n}}^{*} \mathcal{M}_{\alpha \to \gamma_{n}}(2\pi)^{4} \delta^{(4)}(p_{\alpha} - p_{\gamma_{n}}), \\ \text{we have } \operatorname{Im} a_{j}^{\text{el}} \geq \beta_{1} |a_{j}^{\text{el}}|^{2} + |b_{j}^{\text{inel}}|^{2}. \text{ Thus} \\ |b_{j}^{\text{inel}}|^{2} &\leq \frac{1}{4\beta_{1}} - \beta_{1} \left[ (\operatorname{Re} a_{j}^{\text{el}})^{2} + \left(\operatorname{Im} a_{j}^{\text{el}} - \frac{1}{2\beta_{1}}\right)^{2} \right] \leq \frac{1}{4\beta_{1}}. \end{split}$$

For massless incoming particles,

$$\left|b_{j}^{\text{inel}}\right| \leq \frac{1}{2}, \quad \forall j.$$